Journal of Advances in Mathematics and Computer Science

29(1): 1-16, 2018; Article no.JAMCS.43370

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Orbital Motion Around the Collinear Libration Points of the Restricted Three-Body Problem

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Authors contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2018/43370
Editor(s):
(1) Dr. Dragos-Patru Covei, Professor, Department of Applied Mathematics, The Bucharest University of Economic Studies, Piata Romana, Romania.
(2) Dr. Tian-Xiao He, Professor, Department of Mathematics and Computer Science, Illinois Wesleyan University, USA.
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(5) Xuhua Cheng, Hebei University of Technology, China.
Complete Peer review History: http://www.sciencedomain.org/review-history/26393

Received: 07 July 2018
Accepted: 18 September 2018
Published: 25 September 2018

Original Research Article

Abstract

In this work, the special solutions of the Restricted Three-Body Problem (RTBP) are presented which specify the locations of the equilibrium points. The periodic orbits around these libration points are obtained analytically and numerically. The Lissajous orbits around the collinear libration points are focused in this work. Earth-Moon-Spacecraft system is carried out to illustrate this study.

Keywords: Lissajous orbits; periodic orbits; libration points; restricted three-body problem.

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1 Introduction

The three-body problem has been extensively studied during the past two centuries and attracted many authors from Poincare till now. Szebehely made an extensive treatment of the problem [1]. More recently, numerical techniques have been used to generate solutions. Sharma studied periodic orbits of the second kind in the restricted three-body problem when the more massive primary is an oblate spheroid [2]. Sharivastava studied the equations of motion of the restricted problem of three bodies with variable mass [3]. Gabern studied a restricted four-body model for the dynamic near the lagrangian points of the Sun-Jupiter system [4]. Mathlouthi studied the infinity of periodic solution of the restricted three- body problem by using a variational formulation [5]. Llibre studied periodic and Quasi-periodic orbits of the spatial three-body problem [6]. The existence of the periodic orbits near the collinear libration points were treated by many authors. Archambeau [7] determined a class of Eight Lissajous orbits near collinear libration points by using Lindstedt Poincare′s technique. Celletti [8] analyzed the Lissajous and halo orbits near the collinear libration points by using the classical perturbation theory. Ibrahim [9] has determined the libration pints for Sun-Earth-Moon system and has solved the equations of motion at these libration points. Ibrahim [10] studied Lissajous Orbits at the Collinear Libration Points in the Restricted Three-Body Problem with Oblateness.

This paper is organized as follows, the restricted three- body problem is formulated in the equations of motion framework, then the libration points deduced from the equations of motion. The motion around the collinear libration points is treated analytically and numerically. Finally, this study applied on the Earth-Moon system and Sun-Saturn system.

2 Formulating the Restricted Three Body Problem

The motion of an infinitesimal particle influenced by the gravitational force from the central binary can be formulated from Newton′s law of the gravity and the second Newton′s law of the motion. Considering the two primary masses m_1 and m_2 and their positions r_1 and r_2, respectively, where r_1 is the position vector from m_1 to m_3 and r_2 is the position vector from m_2 to m_3, G is Newtonian constant of gravitation, the equations of motion for the third body in the inertial coordinates in the components of x,y and z become:

$$\ddot{x} = -\frac{Gm_1\left(x - x_1\right)}{r_1^3} - \frac{Gm_2\left(x - x_2\right)}{r_2^3} \tag{1}$$

$$\ddot{y} = -\frac{Gm_1\left(y - y_1\right)}{r_1^3} - \frac{Gm_2\left(y - y_2\right)}{r_2^3} \tag{2}$$

$$\ddot{z} = -\frac{Gm_1\left(z - z_1\right)}{r_1^3} - \frac{Gm_2\left(z - z_2\right)}{r_2^3} \tag{3}$$

Let the unit of two masses be chosen and their common center of mass be chosen as $G(m_1+m_2) = 1$, we assume that $m_1 > m_2$ and define $\mu = \frac{m_2}{m_1+m_2}$, so we have:

$$r_1^2 = \left(\mu + x\right)^2 + y^2 + z^2$$

$$r_2^2 = \left(x - (1-\mu)\right)^2 + y^2 + z^2$$

Then in this system of units the two masses are $\mu_1 = Gm_1 = 1 - \mu$, $x_1 = -\mu_2 = -\mu$ and $y_1 = 0$. $\mu_2 = Gm_2 = \mu$, $x_2 = \mu_1 = 1 - \mu$, and $y_2 = 0$.

Equations (1),(2) and (3) yield,

$$\ddot{x} = -\frac{\mu_1\left(x + \mu_2\right)}{r_1^3} - \frac{\mu_2\left(x - \mu_1\right)}{r_2^3} \tag{4}$$

$$\ddot{y} = \frac{(-\mu_1)\,y}{r_1^3} - \frac{\mu_2 y}{r_2^3} = -\left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right] y \tag{5}$$

$$\ddot{z} = -\left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right] z \tag{6}$$

The synodic coordinates are related to the sidereal coordinates by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

The equations of motion of the third body in the rotating system become:

$$\ddot{X} - 2\dot{Y} - X = -\frac{(1-\mu)(\mu+X)}{r_1^3} - \frac{\mu(\mu+X-1)}{r_2^3} \tag{7}$$

$$\ddot{Y} + 2\dot{X} - Y = -\frac{Y(1-\mu)}{r_1^3} - \frac{Y\mu}{r_2^3} \tag{8}$$

$$\ddot{Z} = -\frac{Z(1-\mu)}{r_1^3} - \frac{Z\mu}{r_2^3} \tag{9}$$

The system of Equations (7), (8), and (9) admits an integral of the motion that was originally found by Jacobi [11]:

$$-\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} + \frac{V^2}{2} - \frac{1}{2}\left(X^2 + Y^2 + Z^2\right) = C \tag{10}$$

Where, $\frac{V^2}{2}$ is the kinetic energy per unit mass relative to the rotating frame and $-\frac{\mu_1}{r_1} - \frac{\mu_2}{r_2}$ are the gravitational potential energy of the two primary masses and C is called Jacobi integral, or Jacobi constant.

3 Location of the Libration Points

The libration points are particular solutions to the equations of motion as well as equilibrium solutions. For the equilibrium points,

$$\dot{X} = \dot{Y} = \dot{Z} = 0 \tag{11}$$

$$\ddot{X} = \ddot{Y} = \ddot{Z} = 0 \tag{12}$$

Substituting Equations (11) and (12) into Equations (7), (8) and (9) respectively, this yields,

$$X - \frac{(1-\mu)(\mu+X)}{r_1^3} - \frac{\mu(X-(1-\mu))}{r_2^3} = 0 \tag{13}$$

$$Y - \left[\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right] Y = 0 \tag{14}$$

$$\left[\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}\right] Z = 0 \tag{15}$$

It is well-known that the above system of equations has five equilibrium points. Three of them, denoted by L_1, L_2 and L_3, are collinear with the two primaries and the other two are the triangular equilibrium points of L_4, L_5 where the coordinates of them are [1]

$$L_4\left(\frac{1}{2}-\mu, \frac{\sqrt{3}}{2}\right), \quad L_5\left(\frac{1}{2}-\mu, -\frac{\sqrt{3}}{2}\right) \tag{16}$$

3.1 Location of libration points of L_1, L_2 and L_3

The three collinear points L_1, L_2 and L_3, can be found from Equation (13) [1]. The libration point L_1 lies between masses m_1 and m_2 we can calculate it from nonlinear equation.

$$\frac{\mu}{(xL_1 - (1-\mu))^2} - \frac{1-\mu}{(\mu + xL_1)^2} + xL_1 = 0 \tag{17}$$

which expanded to

$$\begin{aligned} &\mu + 2\mu xL_1 - \mu(xL_1)^2 + (3-2\mu)(xL_1)^3 \\ &\quad + (\mu-3)(xL_1)^4 + (xL_1)^5 = 0 \end{aligned} \tag{18}$$

Where, xL_1 represents the coordinate of L_1. Equation (18) is quintic equation has one real solutions at L_1. Figs. 1 and 2 display the collinear point L_1 for the Earth-Moon system and Sun-Saturn system respectively obtained from the roots of Equation (18).

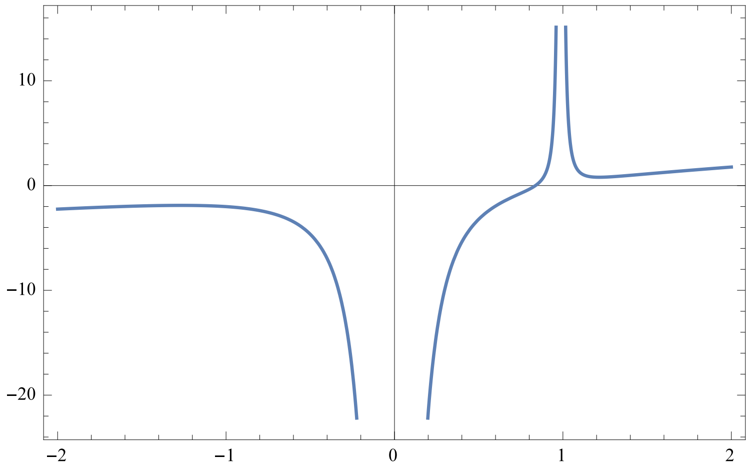

Fig. 1. The libration point L_1 for the Earth-Moon system at L_1= 0.836915

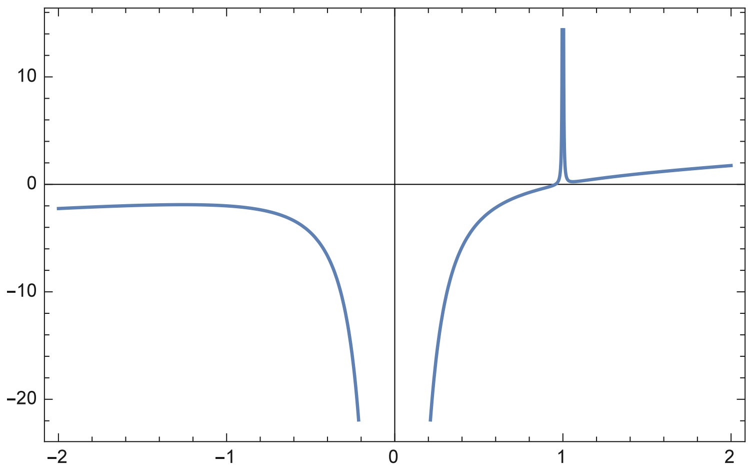

Fig. 2. The libration point L_1 for the Sun-Saturn system atL_1= 0.9547469

The libration point L_2 lies outside the mass m_2 and we can calculate it from nonlinear Equation (19).

$$-\frac{\mu}{(xL_2-(1-\mu))^2}-\frac{1-\mu}{(\mu+xL_2)^2}+xL_2=0 \tag{19}$$

which expanded to

$$\begin{aligned}&-\mu-2\mu xL_2-\mu(xL_2)^2+(3-2\mu)(2xL_2)^3\\&+(3-\mu)(xL_2)^4+(xL_2)^5=0\end{aligned} \tag{20}$$

Where, xL_2 represents the coordinate of L_2. Equation (20) has five roots only one of them is in a real root represents the location of the libration point L_2. while the other roots are imaginary. Figs. 3 and 4 display the collinear point L_2 for the the Earth-Moon system and Sun-Saturn system.

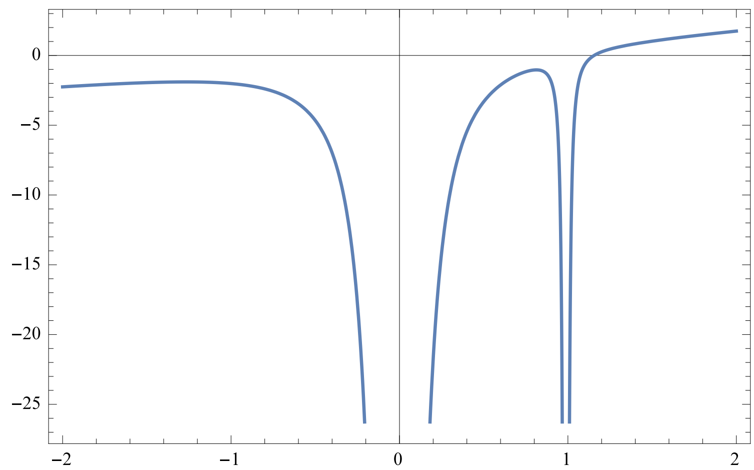

Fig. 3. The libration point L_2 for the Earth-Moon system at L_2=1.15568

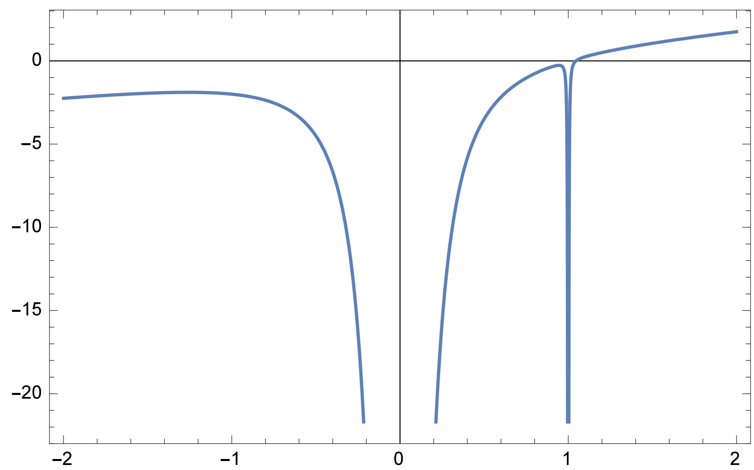

Fig. 4. The libration point L_2 for the Sun-Saturn system at L_2=1.0460716

The libration point L_3 lies on the negative x-axis and we can calculate it from nonlinear Equation(21).

$$-\frac{\mu}{(xL_3-(1-\mu))^2}-\frac{1-\mu}{(\mu+xL_3)^2}+xL_3=0 \tag{21}$$

which expanded to

$$\mu+3(2\mu-2)xL_3+(\mu-1)(xL_3)^2+(2\mu+1)(xL_3)^3 \\ +(\mu+2)(xL_3)^4+(xL_3)^5=0 \tag{22}$$

Where, xL_3 represents the coordinate of L_3. Equation (22) has only one real root which is the location of the libration point L_3. Figs. 5 and 6 display the collinear point L_3 for the Earth-Moon system and the Sun-Saturn system.

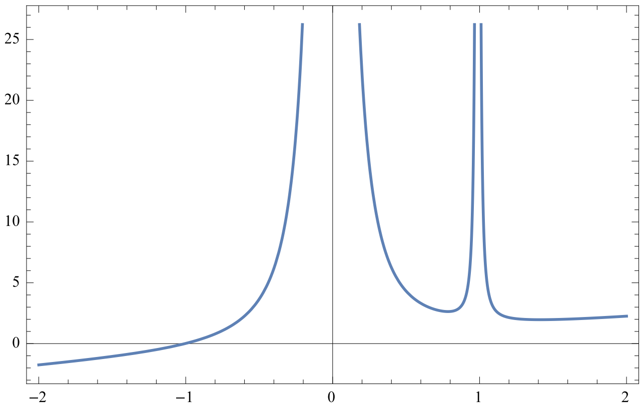

Fig. 5. The libration point L_3 for the Earth-Moon system at L_3 = -1.00506

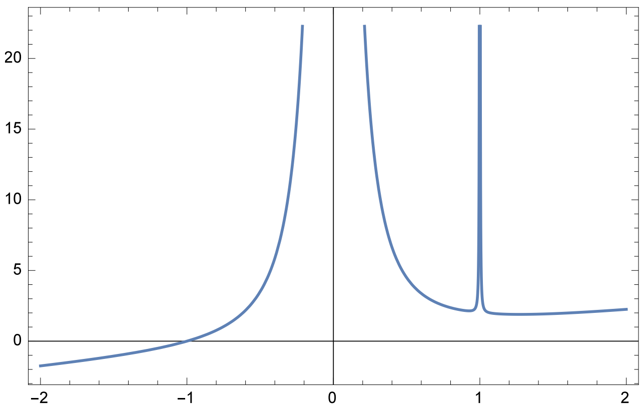

Fig. 6. The libration point L_3 for the Sun-Saturn system at L_3= -1

Finally, Figs. 7 and 8 show the collinear points L_1, L_2 and L_3 on the x-axis for the Earth-Moon and the Sun- Saturn system.

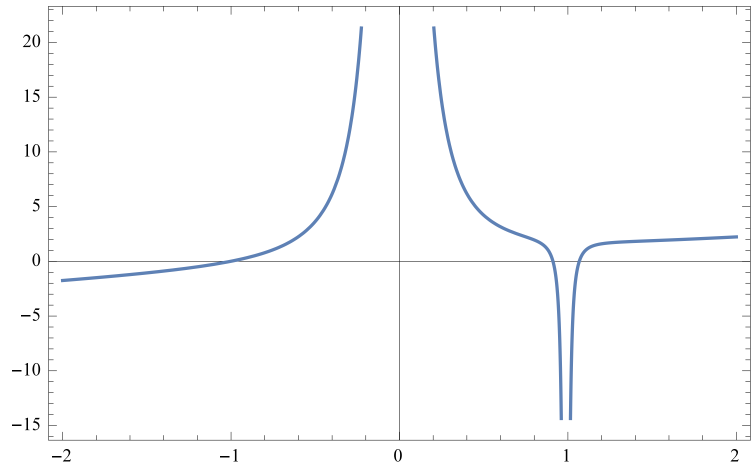

Fig. 7. The collinear points for the Earth-Moon system

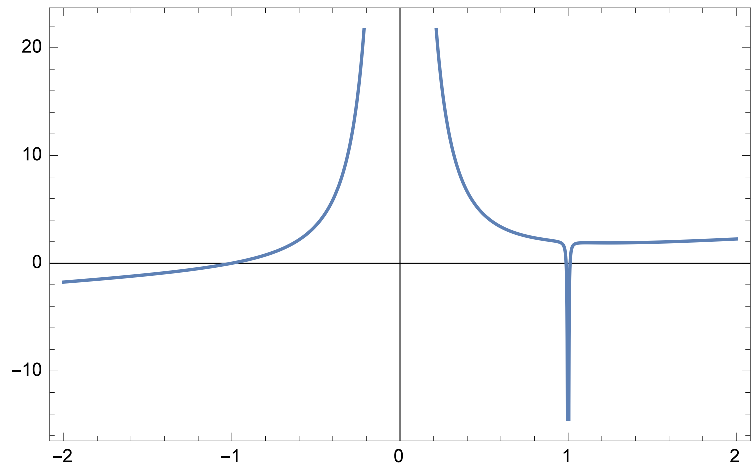

Fig. 8. The collinear points for the Sun- Saturn system

4 The Motion around Collinear Libration Points

The stability of motion near an equilibrium point in the nonlinear system can be obtained by linearizing and producing variationally equations relative to the equilibrium solutions [1]. Thus, a limited investigation of the motion of spacecraft in the vicinity of any libration point can be accomplished with linear analysis. The linear variationally equations associated with libration point Li, corresponding to the position $(xL_i; yL_i; zL_i)$ relative to the barycenter, can be determined through a Taylor series expansion about L_i, retaining only first-order terms. To allow a more compact expression, the variationally variables (ξ, η, ζ) are introduced such that,

$$\xi = x - xL_i, \eta = y - yL_i, \zeta = z - zL_i$$

The resulting linear variationally equations for motion about L_i are written as follows,

$$\ddot{\xi} - 2\dot{\eta} = \xi U_{\mathrm{xx}} + \eta U_{\mathrm{xy}} + \zeta U_{\mathrm{xz}} \tag{23}$$

$$\ddot{\eta} + \dot{\xi} = \eta U_{\mathrm{yy}} + \zeta U_{\mathrm{yz}} + \xi \mathrm{U}_{\mathrm{xy}} \tag{24}$$

$$\ddot{\zeta} = \zeta U_{zz} \tag{25}$$

where,

$$\mathrm{Uxx} = \frac{3\mu(\mu+x-1)^2}{\left((\mu+x-1)^2+y^2+z^2\right)^{5/2}} - \frac{\mu}{\left((\mu+x-1)^2+y^2+z^2\right)^{3/2}} - \frac{1-\mu}{\left((\mu+x)^2+y^2+z^2\right)^{3/2}} + \frac{3(1-\mu)(\mu+x)^2}{\left((\mu+x)^2+y^2+z^2\right)^{5/2}} + 1$$

$$\mathrm{Uyy} = 1 + \frac{3\mu y^2}{\left((\mu+x-1)^2+y^2+z^2\right)^{5/2}} - \frac{\mu}{\left((\mu+x-1)^2+y^2+z^2\right)^{3/2}} - \frac{1-\mu}{\left((\mu+x)^2+y^2+z^2\right)^{3/2}} + \frac{3\left((1-\mu)y^2\right)}{\left((\mu+x)^2+y^2+z^2\right)^{5/2}}$$

$$\mathrm{Uzz} = 1 + \frac{3\mu z^2}{\left((\mu+x-1)^2+y^2+z^2\right)^{5/2}} - \frac{\mu}{\left((\mu+x-1)^2+y^2+z^2\right)^{3/2}} - \frac{1-\mu}{\left((\mu+x)^2+y^2+z^2\right)^{3/2}} + \frac{3\left((1-\mu)z^2\right)}{\left((\mu+x)^2+y^2+z^2\right)^{5/2}}$$

$$\mathrm{Uxy} = \mathrm{Uxy} = \frac{3y(\mu(\mu+x-1))}{\left((\mu+x-1)^2+y^2+z^2\right)^{5/2}} + \frac{3(\mu+x)((1-\mu)y)}{\left((\mu+x)^2+y^2+z^2\right)^{5/2}}$$

$$\mathrm{Uzy} = \mathrm{Uyz} = \frac{3\mu yz}{\left((\mu+x-1)^2+y^2+z^2\right)^{5/2}} + \frac{3y((1-\mu)z)}{\left((\mu+x)^2+y^2+z^2\right)^{5/2}}$$

$$\mathrm{Uzx} = \mathrm{Uxz} = \frac{3z(\mu(\mu+x-1))}{\left((\mu+x-1)^2+y^2+z^2\right)^{5/2}} + \frac{3(\mu+x)((1-\mu)z)}{\left((\mu+x)^2+y^2+z^2\right)^{5/2}}$$

Because all of the libration points are in-plane, the partial derivatives containing z-components are vanished. Therefore, Equations (23) through (25) can be reduced and written as

$$\ddot{\xi} - 2\dot{\eta} = \xi U_{\mathrm{xx}} + \eta U_{\mathrm{xy}} \tag{26}$$

$$\ddot{\eta} + 2\dot{\xi} = \eta U_{\mathrm{yy}} + \xi \mathrm{U}_{\mathrm{xy}} \tag{27}$$

let

$$\xi = Ae^{\lambda \mathrm{t}}, \eta = Be^{\lambda \mathrm{t}} \tag{28}$$

Using the Equation (28) into Equations (26) and (27) then,

$$A\lambda^2 e^{\lambda \mathrm{t}} - 2B\lambda e^{\lambda \mathrm{t}} = Ae^{\lambda \mathrm{t}} U_{\mathrm{xx}} + Be^{\lambda \mathrm{t}} U_{\mathrm{xy}} \tag{29}$$

$$B\lambda^2 e^{\lambda \mathrm{t}} + 2B\lambda e^{\lambda \mathrm{t}} = Ae^{\lambda \mathrm{t}} U_{\mathrm{xy}} + Be^{\lambda \mathrm{t}} U_{\mathrm{yy}} \tag{30}$$

Equations (29) and (30) can be written as,

$$\left(\lambda^2 - U_{\mathrm{xx}}\right) A - B\left(2\lambda + U_{\mathrm{xy}}\right) = 0 \tag{31}$$

$$\left(2\lambda - U_{\mathrm{xy}}\right) A + B\left(\lambda^2 - U_{\mathrm{yy}}\right) = 0 \tag{32}$$

Since A and B are not vanish then,

$$\begin{pmatrix} \lambda^2 - U_{xx} & 2\lambda + U_{xy} \\ 2\lambda - U_{xy} & \lambda^2 - U_{yy} \end{pmatrix} = 0 \tag{33}$$

The determinant of Equation (33) will give the characteristic equation in the form,

$$\lambda^4 + \left(\lambda\left(-U_{xx} - U_{yy} + 4\right)\right)^2 - U_{xy}^2 + U_{xx}U_{yy} = 0 \tag{34}$$

The roots of the characteristic Equation (34) are found to be $\lambda_{1,2} = \pm\lambda$, $\lambda_{3,4} = \pm s$,λ will be real values and s will be imaginary values.

The solutions of the variational Equations (26) and (27) can be expressed in terms of elements depend on time as,

$$\xi = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t} + A_4 e^{\lambda_4 t} \tag{35}$$

$$\eta = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + B_3 e^{\lambda_3 t} + B_4 e^{\lambda_4 t} \tag{36}$$

where A_i and B_i are constant coefficients. and $B_i = \frac{A_i\left(\lambda_i^2 - U_{xx}\right)}{2\lambda_i - U_{xy}} = A_i c_i$,   (i=1,2,3,4)

To obtain the values of constants of integration. Let $\xi_0, \eta_0, \dot{\xi}_0$ and $\dot{\eta}_0$ be the initial coordinates and components of velocity then Equations (35) and (36) give at t = 0,

$$\xi_0 = A_1 + A_2 + A_3 + A_4 \tag{37}$$

$$\dot{\xi}_0 = A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_3 + A_4\lambda_4 \tag{38}$$

$$\eta_0 = c_1(A_1 + A_2) + ic(A_3 + A_4) \tag{39}$$

$$\dot{\eta}_0 = c_1\lambda_1(A_1 + A_2) + isc(A_3 + A_4) \tag{40}$$

By putting $A_1 = A_2 = 0$ to eliminate unstable frequencies λ_1 and λ_2, then the Equations(37) through (40) become,

$$\xi_0 = A_3 + A_4 \tag{41}$$

$$\eta_0 = ic(A_3 + A_4) \tag{42}$$

$$\dot{\eta}_0 = isc(A_3 + A_4) \tag{43}$$

$$\dot{\xi}_0 = A_3\lambda_3 + A_4\lambda_4 \tag{44}$$

$$A_3 = \frac{\xi_0}{2} - \frac{\eta_0 i}{2c} \tag{45}$$

$$A_4 = \frac{\xi_0}{2} + \frac{\eta_0 i}{2c} \tag{46}$$

Where $i = \sqrt{-1}$, $c_i = \frac{\left(\lambda_i^2 - U_{xx}\right)}{2\lambda_i - U_{xy}}$ and $\lambda_{3,4} = \pm s$. Then Equations (35) and (36) become,

$$\begin{aligned} \xi &= \frac{1}{2}\xi_0\left(e^{-ist} + e^{ist}\right) + \frac{(\eta_0 i)\left(e^{ist} - e^{-ist}\right)}{2c} \\ &= \frac{\eta_0 \sin(st)}{c} + \xi_0 \cos(st) \end{aligned} \tag{47}$$

$$\begin{aligned} \eta &= \frac{1}{2}(c\xi_0)\left(e^{ist} - e^{-ist}\right) + \frac{1}{2}i\eta_0\left(e^{-ist} + e^{ist}\right) \\ &= \eta_0 \cos(st) - c\xi_0 \sin(st) \end{aligned} \tag{48}$$

4.1 Lissajous orbits At L_2 for Earth-Moon system

At L_2 $\eta_0 = 0$, then $A_3 = A_4 = A_\xi = \frac{1}{2}\xi_0$

$$\xi(t) = A_\xi \cos(st + \varphi) \tag{49}$$

$$\eta(t) = -A_\xi \sin(st + \varphi) \tag{50}$$

$$\zeta(t) = A_\varsigma \cos(\sigma + tv) \tag{51}$$

Where, φ and σ are the phase angle respectively.

Equations(49),(50) and (51) will be shown the halo and Lissajous orbits around any collinear libration points.

The application of this technique will be show at L_2 of the Earth-Moon system for simplicity. The periodic orbits obtained in the (x,y) plane are called Lyapunov planer orbits as shown in Fig. 9. While the periodic orbits obtained in (x,z) and (y,z) planes are called Lissajous orbits as shown in Figs. 10 and 11 respectively. Tables 1 and 2 display the collinear libration points , the eigen values at each point and Jacobi integral for Earth-Moon system and the Sun-Saturn system respectively.

5 Phase Spaces at Libration Points

To get the periodic orbits about the libration points the following technique will be used. This technique depends on the solution of the system of Equations (7), (8) and (9) taken into account the location of libration point as initial values, it is needed to reduce the order of the differential equations system as follows, let

$$\dot{x}(t) = u(t) \tag{52}$$

$$\dot{y}(t) = v(t) \tag{53}$$

$$\dot{z}(t) = w(t) \tag{54}$$

$$\begin{aligned}\dot{u}(t) = &-(1-\mu)\frac{\mu + x(t)}{r1(t)^3} - \mu\frac{\mu + x(t) - 1}{r2(t)^3} \\ &+ x(t) + 2v(t)\end{aligned} \tag{55}$$

$$\begin{aligned}\dot{v}(t) = &-(1-\mu)\frac{y(t)}{r1(t)^3} - \mu\frac{y(t)}{r2(t)^3} \\ &+ y(t) - 2u(t)\end{aligned} \tag{56}$$

$$\dot{w}(t) = -\frac{(1-\mu)z(t)}{\mathrm{r1}(t)^3} - \frac{\mu z(t)}{\mathrm{r2}(t)^3} \tag{57}$$

The system of differential Equations (52) through (57) can be solved numerically, a code with MATHEMATICA was constructed to solve this system using Implist Runge Kutta method. Fig. 12 displays the phase space for the motion near L_1, also Figs. 13 and 14 display the phase space near L_2 and L_3 for the Earth-Moon-spacecraft system.

Table 1. Earth-Moon libration points at $\mu = 0.0121505816.$ and components of charcterstic roots

Libration points	(XLi , YLi)	(XLi , YLi) km	$\pm\lambda_{1,2}$	$\pm\lambda_{3,4}$	C
L_1	(0.836915, 0)	(334766., 0)	1.43994	1.46698i	3.0009
L_2	(1.15568,0)	(462273., 0)	2.15787	1.86217i	3.00089
L_3	(-1.00506, 0)	($-402025.$, 0)	0.543736	1.41169i	3
L_4	(0.49999, -0.86602)	(334766., 397547.)	0.0045302i	0.99998i	3
L_5	(0.49999, -0.86602)	(334766., $-397547.$)	0.298078i	0.99998i	3

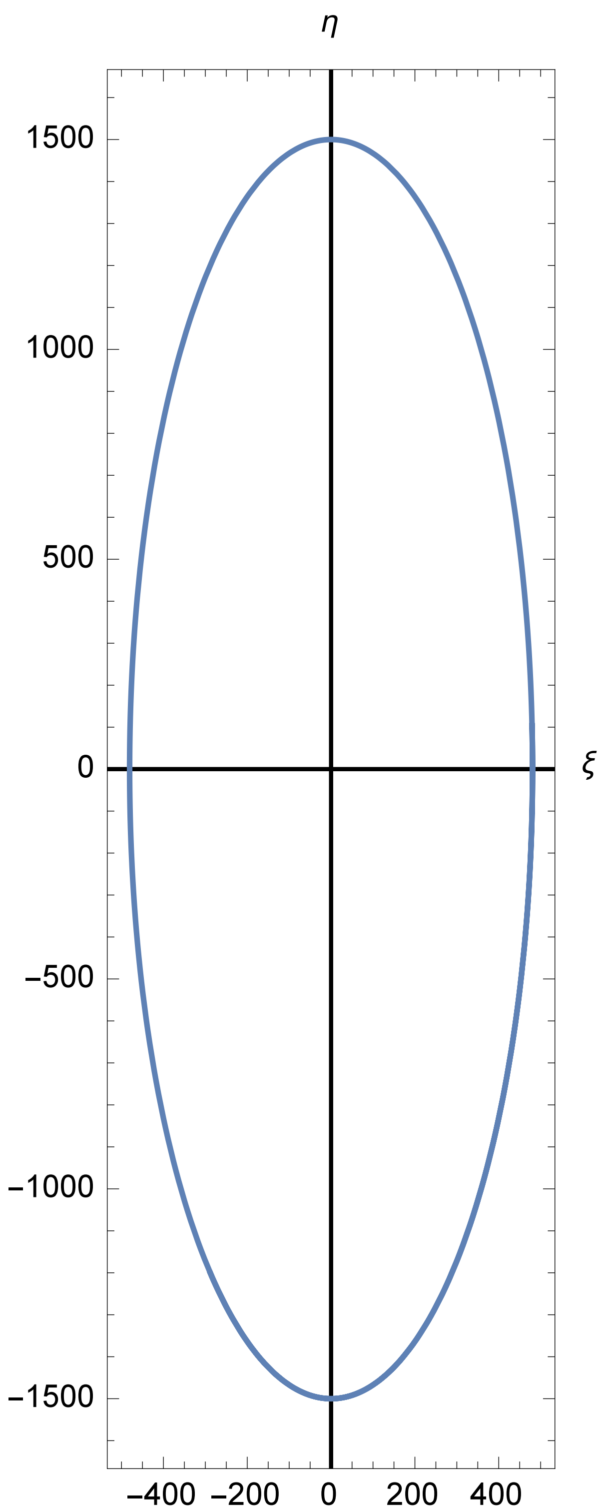

Fig. 9. Lyapunov Orbit around L_2

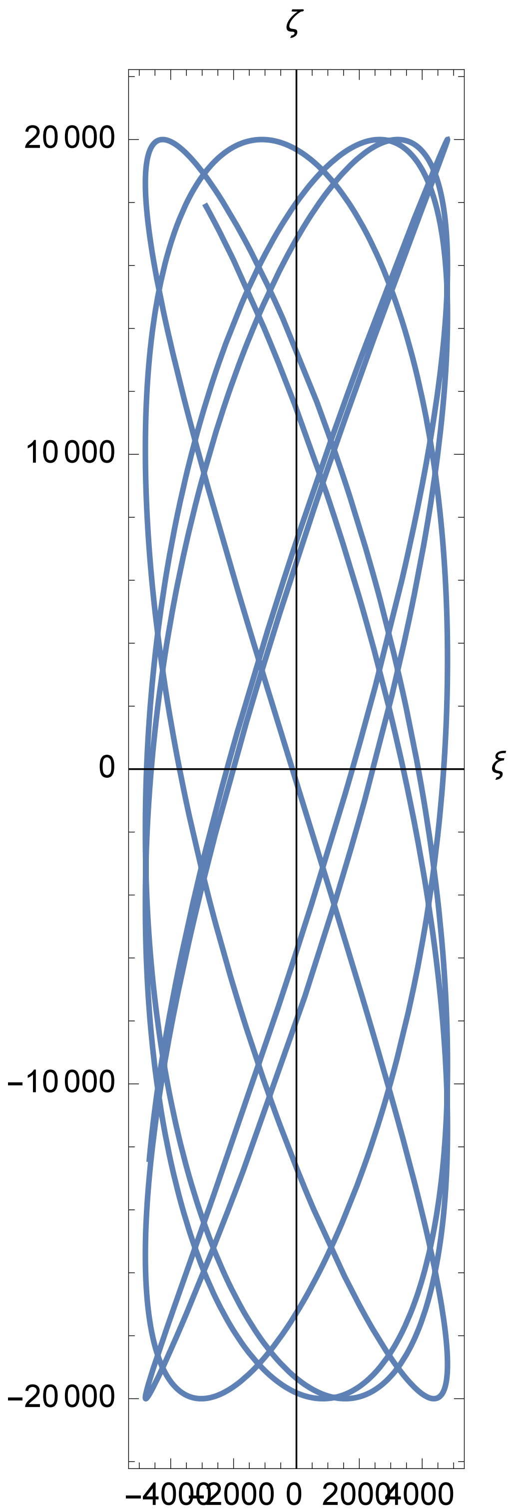

Fig. 10. Lissajous orbits around L_2, ξ against ζ

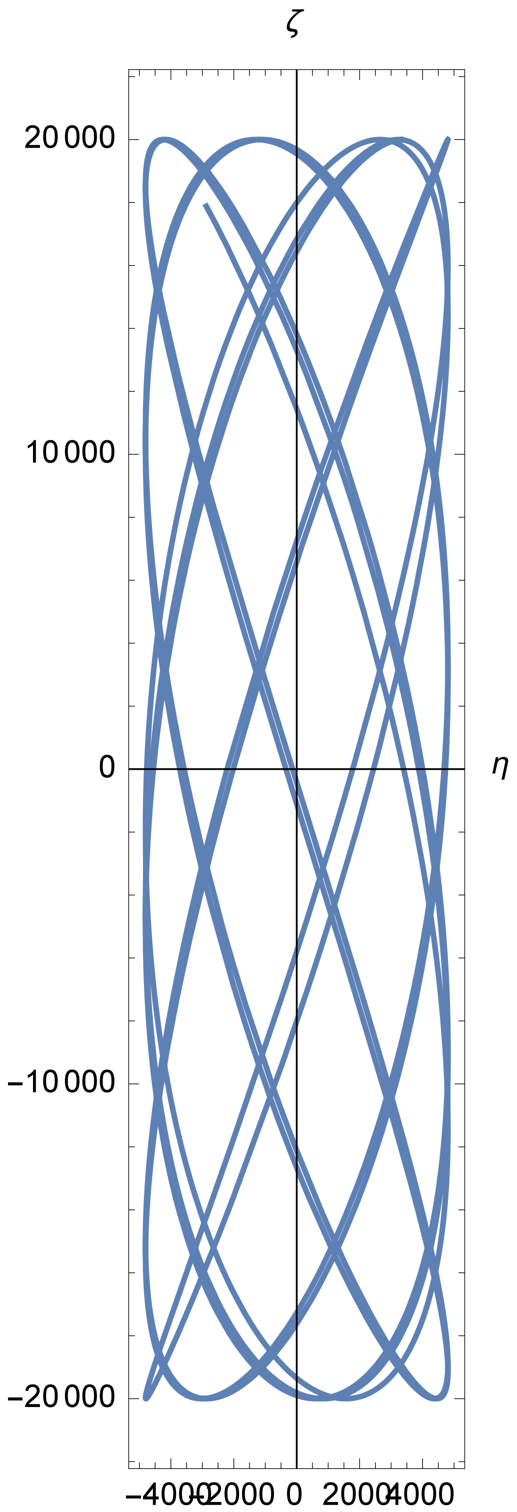

Fig. 11. Lissajous orbits around L_2, η against ζ

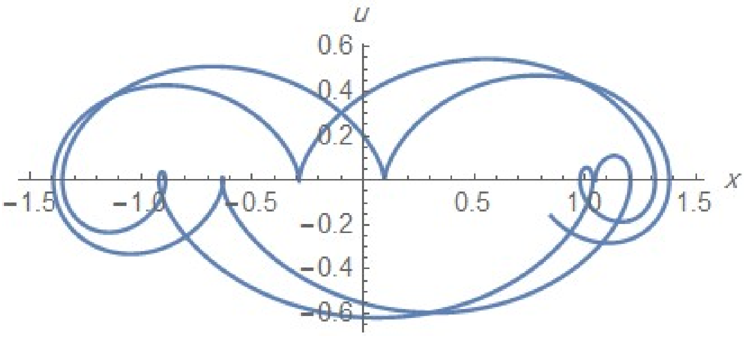

Fig. 12. Phase space about L_1

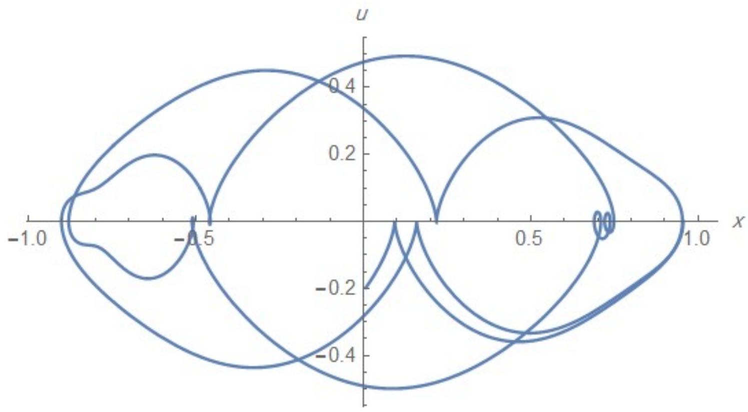

Fig. 13. Phase space about L_2

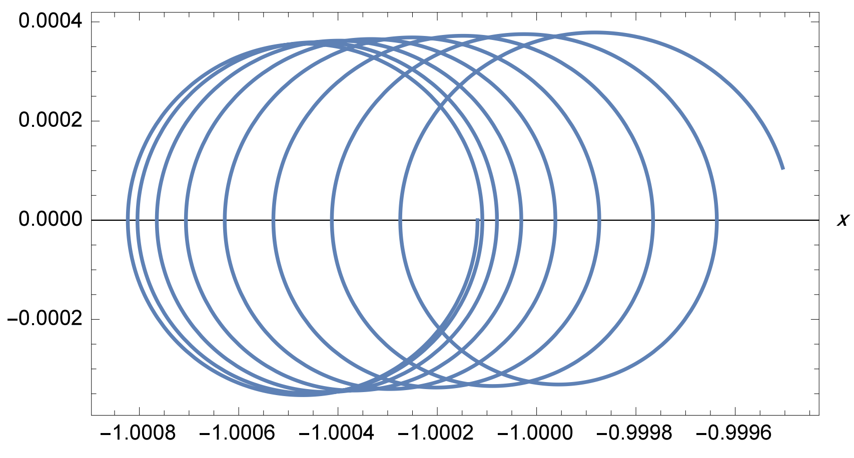

Fig. 14. Phase space at L_3

Table 2. Sun-Saturn libration points at $\mu = 0.0002857696$. and components of charcterstic roots

Libration points	(XLi , YLi)	(Xli , YLi) km	$\pm\lambda_{1,2}$	$\pm\lambda_{3,4}$	C
L_1	(0.9547469, 0)	$(1.36815 * 10^9, 0)$	2.6218685	$2.14112956i$	3.017822
L_2	(1.0460716,0)	$(1.49902 * 10^9, 0)$	2.4075625	$2.0105626i$	3.0174414
L_3	(-1.000119, 0)	$(-1.4331^9, 0)$	0.0273895	$1.0002499i$	3.0002857
L_4	(0.499714, 0.86602)	$(1.36815 * 10^9, 1.43279 * 10^9)$	$0.0439506i$	$0.9990337i$	2.999714
L_5	(0.499714, -0.86602)	$(1.36815 * 10^9, -1.43279 * 10^9)$	$0.0273895i$	$1.0002499i$	2.999714

6 Conclusion

In this study, the equations of motion for the restricted three-body problems were written. The collinear and equatorial libration points for the system were determined. The motion around the collinear libration points were studied. The Lissajous orbits were obtained. The phase spaces about collinear libration points were obtained using Implist Runge-Kutta method. This work is carried out on Earth-Moon system and Sun-Saturn system, which implies periodical motion around the libration points for suitable intervals for the space missions.

Competing Interests

Authors have declared that no competing interests exist.

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