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Type II Half Logistic Rayleigh Distribution: Properties and Estimation Based on Censored Samples

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Authors' contributions

This work was carried out in collaboration between all authors. Author MAH designed the study and performed the statistical analysis. Authors AMA and ASH managed the analyses of the study, wrote the protocol, wrote the first draft of the manuscript. Author ME managed the literature searches. All authors read and approved the final manuscript.

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Abstract

In this paper, we introduce a new model called the *Type II half logistic Rayleigh* (TIIHLR) with U-shaped, increasing and decreasing hazard rate function. Some structural properties of the current distribution are derived including; explicit expressions for moments, incomplete moments, order statistics and Rényi entropy. Maximum likelihood estimators of the model parameters, based on complete and censored samples, are obtained. A numerical study is demonstrated to illustrate the theoretical results. The superiority of the new model over some new existing distributions is illustrated through two real data sets. In both applications, the TIIHLR model produces better fits than; the transmuted Rayleigh; transmuted generalised Rayleigh; exponentiated transmuted generalised Rayleigh and transmuted exponentiated inverse Rayleigh distributions.

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1 Introduction

Most of the popular traditional distributions often do not characterise and do not predict the real world phenomena. However, the usual distributions sometimes have some respective drawbacks in analysing lifetime data. So, many generalized classes of distributions have been developed and applied to describe most of the interesting data sets. In the last few years, the generated family of continuous distributions is a new improvement for producing and extending the usual classical distributions. These families have been broadly studied in several areas as well as yield more flexibility in many applications. Some of the generators are: Beta-generated (B-G) (Eugene et al. [1]), Gamma-G (Zografos and Balakrishnan [2]) and Ristic and Balakrishnan [3]), Kumaraswamy-G (Cordeiro and de Castro [4]), exponentiated generalized class (Cordeiro et al. [5]), Weibull-G (Bourguignon et al. [6]), Garhy-G (Elgarhy et al. [7]), Kumaraswamy Weibull-G (Hassan and Elgarhy [8]), exponentiated Weibull-G (Hassan and Elgarhy [9]), additive Weibull-G (Hassan and Hemeda [10]), exponentiated extended-G (Elgarhy et al. [11], Type II half logistic-G(TIIHL-G) (Hassan et al. [12]), generalized additive Weibull-G (Hassan et al. [13]), odd Frechet-G (Haq and Elgarhy [14]), power Lindley-G (Hassan and Nassr [15] and Muth-G (Almarashi and Elgarhy [16]) among others.

According to Hassan et al. [12], the *cumulative distribution function* (cdf) and *the probability density function* (pdf) of TIIHL-G family of distributions are defined, respectively, as follows

$$F(x;\lambda,\zeta)=\frac{2\left[G(x;\zeta)\right]^{\lambda}}{1+\left[G(x;\zeta)\right]^{\lambda}},x>0\ ,\ \lambda>0, \tag{1}$$

and,

$$f\left(x;\lambda,\zeta\right)=\frac{2\lambda g(x;\zeta)\left[G(x;\zeta)\right]^{\lambda-1}}{\left[1+\left[G(x;\zeta)\right]^{\lambda}\right]^{2}},x>0\ ,\ \lambda>0, \tag{2}$$

where, λ is the shape parameter and $G(x;\zeta)$ is the baseline distribution, which depends on ζ. However different values to $G(x;\zeta)$ give new distributions.

In reliability study, life-tests are performed to observe the life of the experimental units put on a test. In such test, some surviving units are removed or lost due to time and cost constraints or due to immediate needs of the units for other purposes. The data obtained from such a life-test are generally censored samples. The most common censoring schemes are Type-I and Type-II. In Type-I censoring scheme, the experiment continues until a pre-assigned time *T*, and failures that occur after *T* are not observed. In contrast, in Type-II censoring scheme the experiment decides to terminate the test after a pre-assigned number of failures observed, say k, $k\leq n$.

Our aim in this work is to introduce and study a new two-parameter lifetime model, depending on Rayleigh distribution to increase its flexibility for various modelling purposes. Further, estimation of the population parameters in case of complete and censored samples is discussed. This paper can be sorted as follows. In the next section, the TIIHLR distribution is defined. Section 3 concerns some general mathematical properties of the TIIHLR distribution. The maximum likelihood method is applied to obtain the estimators of the model parameters, and the simulation study is provided in Section 4. Based on Type I and Type II censored samples; maximum likelihood estimators of the model parameters and simulation issues are obtained in Section 5. An illustrative purpose on the basis of real data is investigated, in Section 6. Finally, concluding remarks are handled in Section 7.

2 Type II Half Logistic Rayleigh Distribution

The Rayleigh distribution plays a vital role in modelling the lifetime in many practical applications, including reliability, life testing and survival analysis. Rayleigh is a special case from the well-known Weibull distribution. The cdf of Rayleigh distribution with scale parameter δ is defined by

$$G(x;\delta)=1-e^{-\delta x^2}, \qquad \delta, x>0. \tag{3}$$

Several authors have discussed estimation of the model parameters for Rayleigh distribution; see for example; Howlader and Hossian [17], Abd Elfattah et al. [18,19], Lalitha and Mishra [20], Hendi et al. [21], Dey and Das [22], Dey [23].

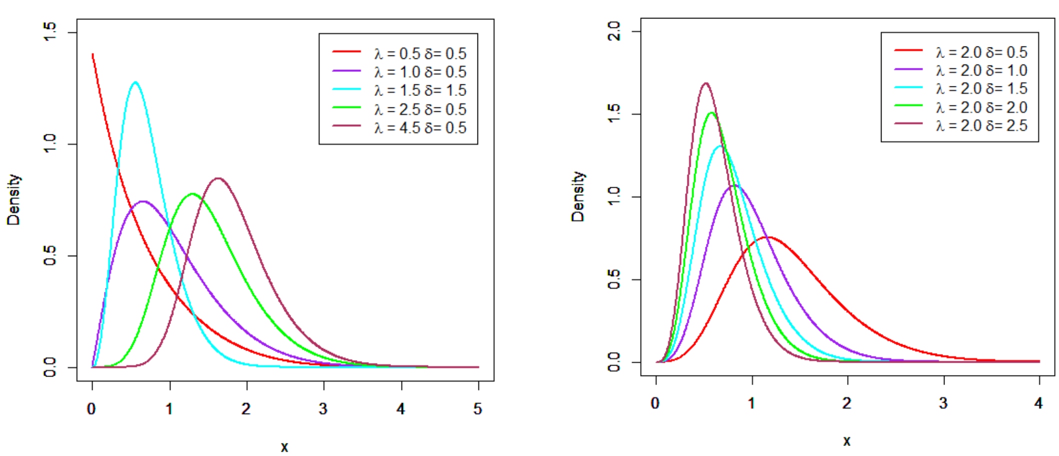
In this section, we obtain the pdf, cdf, survival and hazard rate, cumulative hazard rate, reversed hazard rate and odds ratio functions of TIIHLR distribution. The cdf of the TIIHLR distribution is obtained by substituting cdf (3) in (1) as follows

$$F(x;\lambda,\delta)=\frac{2\left[1-e^{-\delta x^2}\right]^{\lambda}}{1+\left[1-e^{-\delta x^2}\right]^{\lambda}}, \qquad \lambda,\delta>0 \quad , \quad x>0. \tag{4}$$

The pdf corresponding to (4) is given by

$$f(x;\lambda,\delta)=\frac{4\lambda\delta x e^{-\delta x^2}[1-e^{-\delta x^2}]^{\lambda-1}}{\left[1+[1-e^{-\delta x^2}]^{\lambda}\right]^2}. \tag{5}$$

Plots of a random variable X with density function (5) can be represented through Fig. 1. As it seems from Fig. 1, that the pdf of TIIHLR can take different shapes according to different values of λ and δ. It can be symmetric, right skewed, unimodel and reversed J-shaped.



(a) (b)

Fig. 1. The pdf plots of TIIHLR distribution for different values of parameters

The survival function, say $\overline{F}(x;\lambda,\delta),$ is given by

$$\overline{F}(x;\lambda,\delta)=\frac{1-\left[1-e^{-\delta x^2}\right]^{\lambda}}{1+\left[1-e^{-\delta x^2}\right]^{\lambda}}.$$

The *hazard rate function* (hrf), say $h(x;\lambda,\delta),$ is as follows

$$h(x;\lambda,\delta)=\frac{4\lambda\delta x e^{-\delta x^2}[1-e^{-\delta x^2}]^{\lambda-1}}{1-[1-e^{-\delta x^2}]^{2\lambda}},$$

Plots of hrf for the TIIHLR distribution are displayed in Fig. 2. It can be deduced from Fig. 2 that the shape of the hrf of the TIIHLR distribution can be constant, increasing or decreasing and U-shaped (depending on the value of the parameters). This indicates that The TIIHLR has more flexibility than the classical Rayleigh model.

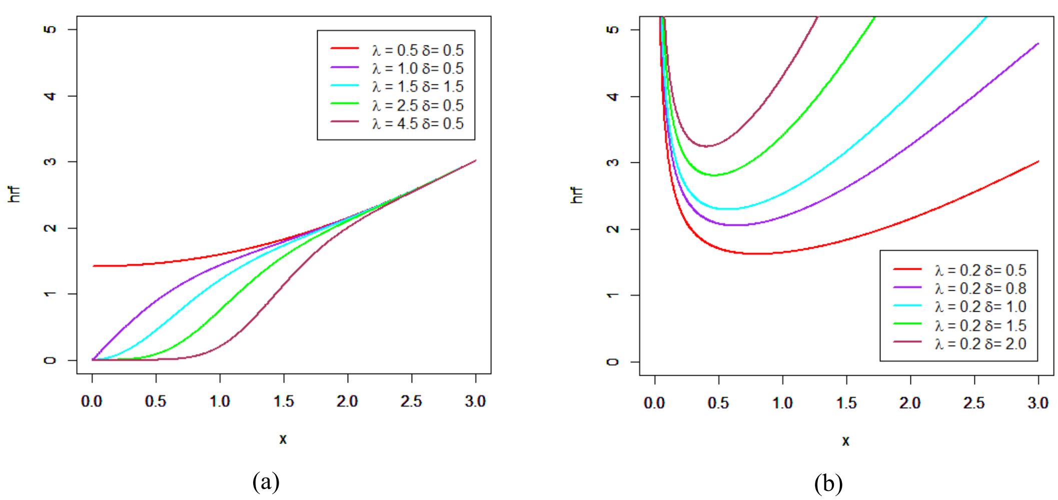


Fig. 2. The hrf plots of TIIHLR distribution for different values of parameters

The reversed hazard rate function, say $r(x;\lambda,\delta),$ is given by

$$r(x;\lambda,\delta)=\frac{f(x;\lambda,\delta)}{F(x;\lambda,\delta)}=\frac{2\lambda\delta x e^{-\delta x^2}}{\left[1+[1-e^{-\delta x^2}]^{\lambda}\right]^2\left[1-e^{-\delta x^2}\right]}.$$

Furthermore, the cumulative hazard rate, say $H(x;\lambda,\delta),$ and the odds function; say $O(x;\lambda,\delta),$ are, respectively, given by

$$H(x;\lambda,\delta) = -\ln(\overline{F}(x;\lambda,\delta)) = -\ln\left[\frac{1-\left[1-e^{-\delta x^2}\right]^{\lambda}}{1+\left[1-e^{-\delta x^2}\right]^{\lambda}}\right],$$

and,

$$O(x;\lambda,\delta) = \frac{F(x;\lambda,\delta)}{\overline{F}(x;\lambda,\delta)} = \frac{2}{\left[1-e^{-\delta x^2}\right]^{-\lambda}-1}.$$

3 Structural Properties

Here, we provide some statistical properties of TIIHLR distribution.

3.1 Moments

Some of the most important characteristics and merit of any distribution can be studied through its moments. The *i* moment for the TIIHLR distribution about zero is obtained from (5) as follows

$$E(X^r) = \int_0^{\infty} x^r f(x;\lambda,\delta)dx = \int_0^{\infty} x^r \frac{4\lambda\delta x e^{-\delta x^2}[1-e^{-\delta x^2}]^{\lambda-1}}{\left[1+[1-e^{-\delta x^2}]^{\lambda}\right]^2}dx. \tag{6}$$

Since, the generalised binomial series expansion is as follows

$$(1+z)^{-\beta} = \sum_{i=0}^{\infty}(-1)^i \binom{\beta+i-1}{i} z^i, \qquad |z|<1, \beta>0. \tag{7}$$

Then applying (7) in (6), then (6) can be expressed as follows

$$E(X^r) = \int_0^{\infty} x^r \sum_{i=0}^{\infty} 4\lambda(-1)^i (i+1)\delta x e^{-\delta x^2}[1-e^{-\delta x^2}]^{\lambda(i+1)-1}dx. \tag{8}$$

Also, it is known that

$$(1-z)^{\beta-1} = \sum_{j=0}^{\infty}(-1)^j \binom{\beta-1}{j} z^j. \tag{9}$$

Hence, by applying (9) in (8), we obtain

$$E(X^r) = \sum_{i,j=0}^{\infty}(-1)^{i+j}\ (i+1)4\lambda\delta \binom{\lambda(i+1)-1}{j} \int_0^{\infty} x^{r+1} e^{-\delta(j+1)x^2}dx. \tag{10}$$

Hence, after some manipulation, the r^{th} moment of TIIHLR distribution takes the following form

$$E(X^r)=\sum_{i,j=0}^{\infty}\eta_{i,j}\frac{\Gamma(\frac{r}{2}+1)}{\left[\delta(j+1)\right]^{\frac{r}{2}+1}};\qquad r=1,2,...$$

where

$$\eta_{i,j}=(-1)^{i+j}\,2\lambda\delta\,(i+1)\binom{\lambda(i+1)-1}{j}$$

Furthermore, for a random variable *X*, it is known that the moment generating function is defined as

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r).$$

So, the moment generating function of TIIHLR distribution takes the following form

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r)=\sum_{r,i,j=0}^{\infty}\frac{\eta_{i,j}t^r\Gamma(\frac{r}{2}+1)}{r!\left[\delta(j+1)\right]^{\frac{r}{2}+1}}.$$

3.2 Incomplete moments

The s^{th} incomplete moment of the TIIHLR distribution, say ($\mathrm{E}_s(t)$), is obtained by using pdf (5) as follows

$$\mathrm{E}_s(t)=\int_0^t x^s\,\mathrm{f}(x;\lambda,\delta)dx=\int_o^t x^s\frac{4\lambda\delta x e^{-\delta x^2}[1-e^{-\delta x^2}]^{\lambda-1}}{\left[1+[1-e^{-\delta x^2}]^{\lambda}\right]^2}dx. \tag{11}$$

Hence, by applying binomial expansions (7) and (9) in (11), then we obtain

$$\mathrm{E}_s(t)=\sum_{i,j=0}^{\infty}(-1)^{i+j}\,(i+1)4\lambda\delta\binom{\lambda(i+1)-1}{j}\int_0^t x^{s+1}e^{-\delta(j+1)x^2}dx.$$

After some simplification, the s^{th} moment of TIIHLR distribution takes the following form

$$\mathrm{E}_s(t)=\sum_{i,j=0}^{\infty}\frac{\eta_{i,j}}{\delta(j+1)^{\frac{s}{2}+1}}\int_0^{\delta(j+1)t^2}y^{\frac{s}{2}}e^{-y}dy;$$

which is the lower incomplete gamma function. Hence, the s^{th} incomplete moment of the TIIHLR distribution takes the following form

$$\mathrm{E}_s(t)=\sum_{i,j=0}^{\infty}\frac{\eta_{i,j}}{\delta(j+1)^{\frac{s}{2}+1}}\gamma(\frac{s}{2}+1,\delta(j+1)t^2),\qquad s=1,2,... \tag{12}$$

where γ (.,.). is the lower incomplete gamma function.

Bonferroni and Lorenz curves are important applications for the first incomplete moments. These curves are useful in economics, reliability, demography, insurance and medicine The Lorenz and Bonferroni curves are obtained, respectively, as follows

$$L_F(x)=\frac{1}{E(X)}\int_0^x t\ f(t)\ dt=\frac{2\sum_{i,j=0}^{\infty}\frac{\eta_{i,j}}{(\delta(j+1))^{\frac{3}{2}}}\gamma(\frac{3}{2},\delta(j+1)x^2)}{\sum_{i,j=0}^{\infty}\frac{\eta_{i,j}\sqrt{\pi}}{\left[\delta(j+1)\right]^{\frac{3}{2}}}},$$

and

$$B_F(x)=\frac{L_F(x)}{F(x;\lambda,\delta)}=\frac{1+\left[1-e^{-\delta x^2}\right]^{\lambda}\sum_{i,j=0}^{\infty}\frac{\eta_{i,j}}{\left[\delta(j+1)\right]^{\frac{3}{2}}}\gamma(\frac{3}{2},\delta(j+1)x^2)}{\left[1-e^{-\delta x^2}\right]^{\lambda}\sum_{i,j=0}^{\infty}\frac{\eta_{i,j}\sqrt{\pi}}{\left[\delta(j+1)\right]^{\frac{3}{2}}}}.$$

The mean deviations provide useful information about the characteristics of a population. The mean deviations of X about the mean (μ) and about the median (m) can be calculated from the following relations

$$\delta_1=2\mu F(\mu)-2T(\mu)\qquad \text{and } \delta_2=\mu-2T(m)$$

where , $T(\mu)$ and $T(m)$ are the first incomplete moments, which are obtained from (12) as follows;

$$T(\mu)=\int_0^{\mu}x\,f(x)\,dx=\sum_{i,j=0}^{\infty}\frac{\eta_{i,j}}{\left[\delta(j+1)\right]^{\frac{3}{2}}}\gamma(\frac{3}{2},\delta(j+1)\mu^2),$$

and,

$$T(m)=\int_0^{m}x\ f(x)\ dx=\sum_{i,j=0}^{\infty}\frac{\eta_{i,j}}{\left[\delta(j+1)\right]^{\frac{3}{2}}}\gamma(\frac{3}{2},\delta(j+1)m^2).$$

3.3 Rényi entropy

The entropy of a random variable X is a measure of variation of uncertainty and has been used in many fields such as physics, engineering and economics. Rényi [24] defined the Rényi entropy as follows

$$I_{\omega}(X)=\frac{1}{1-\omega}\log\int_{-\infty}^{\infty} f(x)^{\omega}\,dx, \quad \omega>0 \text{ and } \omega\neq 1. \tag{13}$$

Substituting pdf (5) in (13) and applying the binomial theory (7) and (9), then the pdf $f\,(x\,;\lambda,\delta)^{\omega}$ can be expressed as follows

$$f\,(x\,;\lambda,\delta)^{\omega}=\sum_{i,j=0}^{\infty} t_{i,j}\, x^{\omega} e^{-\delta(w+j)x^{2}},$$

where

$$t_{i,j}=(4\lambda\delta)^{\omega}(-1)^{i+j}\binom{2\omega+i-1}{i}\binom{\lambda(\omega+i)-\omega}{j}.$$

Therefore, the Rényi entropy of TIIHLR distribution is given by

$$I_{\omega}(X)=\frac{1}{1-\omega}\log\left[\sum_{i,j=0}^{\infty}\frac{t_{i,j}\,\Gamma(\frac{\omega+1}{2})}{2\left[\delta(\omega+j)\right]^{\frac{\omega+1}{2}}}\right].$$

3.4 Order statistics

Order statistics have been extensively applied in many fields of statistics, such as reliability and life testing. Let X_1, X_2,…, X_n be independent and identically distributed random variables with their corresponding continuous distribution function. Let $X_{(1)}< X_{(2)}<\ldots< X_{(n)}$ be the corresponding ordered random sample from a population of size n. The pdf of the r^{th} order statistic is given by

$$f_{(r)}(x\,;\lambda,\delta)=\frac{f\,(x\,;\lambda,\delta)}{B\,(r,n-r+1)}\sum_{v=0}^{n-r}(-1)^{v}\binom{n-r}{v}F\,(x\,;\lambda,\delta)^{v+r-1}, \tag{14}$$

$B(.,.)$is the beta function. The pdf of the r^{th} order statistic for TIIHLR distribution is derived by substituting (4) and (5) in (14) as follows

$$f_{(r)}(x;\lambda,\delta)=\frac{1}{B(r,n-r+1)}\sum_{v=0}^{n-r}(-1)^{v}\binom{n-r}{v}\frac{2^{v+r+1}\lambda\delta x e^{-\delta x^{2}}[1-e^{-\delta x^{2}}]^{\lambda(v+r)-1}}{\left[1+[1-e^{-\delta x^{2}}]^{\lambda}\right]^{v+r+1}}. \tag{15}$$

Applying the binomial expansion (7) in (15), then we have

$$f_{(r)}(x;\lambda,\delta)=\frac{1}{B(r,n-r+1)}\sum_{v=0}^{n-r}\sum_{i=0}^{v+r-1}(-1)^{v+i}\binom{n-r}{v}\binom{v+r+i}{i}2^{v+r+1}\lambda\delta x e^{-\delta x^2}[1-e^{-\delta x^2}]^{\lambda(v+r+i)-1}.$$

Again, using the binomial expansion (9) in the previous equation, then the pdf of the r^{th} order statistic for TIIHLR distribution is obtained as follows

$$f_{(r)}(x;\lambda,\delta)=\frac{1}{B(r,n-r+1)}\sum_{v=0}^{n-r}\sum_{i=0}^{v+r-1}\sum_{j=0}^{\infty}\eta^{*}\lambda\delta x e^{-\delta(j+1)x^2}, \tag{16}$$

$$\eta^{*}=(-1)^{v+i+j}\,2^{v+r+1}\binom{n-r}{v}\binom{v+r+i}{i}\binom{\lambda(v+r+i)-1}{j}.$$

Individually, the distribution of the smallest and largest order statistics are obtained by putting $r=1$ and $r=n$ in (16) respectively as follows

$$f_{(1)}(x;\lambda,\delta)=n\sum_{v=0}^{n-1}\sum_{i=0}^{v+r-1}\sum_{j=0}^{\infty}\eta^{**}\lambda\delta x e^{-\delta(j+1)x^2},$$

$$\eta^{**}=(-1)^{v+i+j}\,2^{v+2}\binom{n-1}{v}\binom{v+1+i}{i}\binom{\lambda(v+1+i)-1}{j}.$$

and,

$$f_{(n)}(x;\lambda,\delta)=n\sum_{i=0}^{v+n-1}\sum_{j=0}^{\infty}\eta^{***}\lambda\delta x e^{-\delta(j+1)x^2},$$

$$\eta^{***}=(-1)^{v+i+j}\,2^{v+n+1}\binom{v+n+i}{i}\binom{\lambda(v+n+i)-1}{j}.$$

4 Parameter Estimation Based on Complete Samples

In this section, we obtain the maximum likelihood estimators of TIHLR distribution in case of complete samples and simulation study is performed.

4. 1 Maximum likelihood estimators

This subsection deals with the maximum likelihood estimators of the unknown parameters for the TIIHLR distribution. Let $X_1, X_2, \ldots, X_n$ be the observed values from the TIIHLR distribution The log-likelihood function of TIIHLR with pdf (5), denoted by $\ln L$, is obtained as follows

$$\ln L = n\ln 2\lambda + n\ln\delta + n\ln 2 + \sum_{i=1}^{n}\ln\left(x_i\right) - \delta\sum_{i=1}^{n}x_i^{\,2} \\ +(\lambda-1)\sum_{i=1}^{n}\ln\left[1-e^{-\delta x_i^{\,2}}\right] - 2\sum_{i=1}^{n}\ln\left[1+\left[1-e^{-\delta x_i^{\,2}}\right]^{\lambda}\right]. \tag{17}$$

Considering the two parameters λ and δ are unknown and differentiating the log-likelihood function (17) with respect to λ and δ as follows

$$\frac{\partial \ln L}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\ln\left[1-e^{-\delta x_i^{\,2}}\right] - 2\sum_{i=1}^{n}\frac{\ln\left[1-e^{-\delta x_i^{\,2}}\right]}{\left[\left[1-e^{-\delta x_i^{\,2}}\right]^{-\lambda}+1\right]},$$

and

$$\frac{\partial \ln L}{\partial\delta} = \frac{n}{\delta} - \sum_{i=1}^{n}x_i^{\,2} + (\lambda-1)\sum_{i=1}^{n}\frac{x_i^{\,2}e^{-\delta x_i^{\,2}}}{1-e^{-\delta x_i^{\,2}}} - 2\lambda\sum_{i=1}^{n}\frac{x_i^{\,2}e^{-\delta x_i^{\,2}}[1-e^{-\delta x_i^{\,2}}]^{\lambda-1}}{1+\left[1-e^{-\delta x_i^{\,2}}\right]^{\lambda}}.$$

Setting $\partial\ln L/\partial\lambda$ and $\partial\ln L/\partial\delta$ equal to zero and solving these equations numerically yield the maximum likelihood estimate (MLE).

4.2 Simulation study

In this subsection, an extensive numerical investigation will be carried out to evaluate the performance of MLE for TIIHLR model. Performance of estimators is evaluated through their biases and *mean square errors* (MSEs) for different sample sizes. A numerical study is performed using Mathematica (7) software.

Different sample sizes are generated through the experiments at sample size n =10, 20, 30, 50 and 100. The generation of TIIHLR distribution is very simple, if U has a uniform (0,1) random number, then

$$X_i = \sqrt{\frac{-1}{\delta}\ln\left(1-\left[\frac{u_i}{2-u_i}\right]^{\frac{1}{\lambda}}\right)}, i = 1,2,\ldots n;$$

follows TIIHLR distribution.

In addition, the different values of parameters λ and δ are considered as λ =0.5 and δ =0.5,0.9 1.2 and 1.5 . The experiment will be repeated 10000 times.

For each sample size and for selected values of parameters, the MLEs of the model parameters are obtained. Hence; the MSEs and biases for the different estimators are recorded in Tables (1) and (2)

Table 1. MLEs, Biases, MSEs of TIIHLR distribution

		(0.5,0.5)			**(0.5,0.9)**		
n		**MLE**	**Bias**	**MSE**	**MLE**	**Bias**	**MSE**
	λ	0.5880	0.0880	0.0901	0.5878	0.0878	0.0872
10	δ	0.5145	0.0145	0.0080	0.9279	0.4279	0.2081
	λ	0.5382	0.0382	0.0267	0.5397	0.0397	0.0272
20	δ	0.5089	0.0089	0.0042	0.9144	0.4144	0.1848
	λ	0.5235	0.0235	0.0156	0.5260	0.0260	0.0158
30	δ	0.5064	0.0064	0.0028	0.9093	0.4093	0.1761
	λ	0.5164	0.0164	0.0084	0.5144	0.0144	0.0083
50	δ	0.5032	0.0032	0.0017	0.9059	0.4059	0.1702
100	λ	0.5057	0.0057	0.0038	0.5079	0.0079	0.0040
	δ	0.5018	0.0018	0.0008	0.9025	0.4025	0.1647

Table 2. MLEs, Biases, MSEs of TIIHLR distribution

		(0.5,1.2)			**(0.5,1.5)**		
n		**MLE**	**Bias**	**MSE**	**MLE**	**Bias**	**MSE**
	λ	0.5875	0.0875	0.0858	0.5943	0.0943	0.0982
10	δ	1.2356	0.7356	0.5854	1.5462	1.0462	1.1642
	λ	0.5384	0.0384	0.0286	0.5433	0.0433	0.0282
20	δ	1.2200	0.7200	0.5416	1.5216	1.0216	1.0793
	λ	0.5267	0.0267	0.0161	0.5269	0.0269	0.0158
30	δ	1.2111	0.7111	0.5211	1.5157	1.0157	1.0565
	λ	0.5135	0.0135	0.0083	0.5165	0.0165	0.0084
50	δ	1.2096	0.7096	0.5130	1.5084	1.0084	1.0313
	λ	0.5063	0.0063	0.0038	0.5074	0.0074	0.0039
100	δ	1.2061	0.7061	0.5034	1.5038	1.0038	1.0152

The values in the preceding tables show that the MSE for the estimates of the parameters λ and δ decreases as the sample size increases. The MSEs of λ estimates are smaller than the corresponding MSEs of δ estimate for different sample sizes.

5 Parameter Estimation Based on Censoring Samples

In reliability or lifetime testing experiments, most of the data are censored due to various reasons such as time limitation, cost or other resources. Here we discuss the estimation of population parameters of TIIHLR distribution based on two censoring schemes; namely, Type I and Type II. In Type-I censoring, we have a fixed time say T but the number of items fails during the experiment is random. Whereas, in Type-II censoring scheme, the experiment is continued (i.e. time varies) until the specified number of failures k occur.

5.1 Maximum likelihood estimators in case of type-I censored samples

Suppose that n items, whose life times follow TIIHLR distribution (5) are placed on a life test, and the test is terminated at a specified time T before all n items have failed. The number of failures k and all failure times are random variables. The log-likelihood function, based on Type-I censoring, is given by:

$$\ln l_1 = \ln\left(\frac{n!}{(n-k)!}\right) + k\ln 2\lambda + k\ln\delta + k\ln 2 + \sum_{i=1}^{k}\ln\left(x_{(i)}\right) - \delta\sum_{i=1}^{k}x_{(i)}^{2} + (\lambda-1)\sum_{i=1}^{k}\ln\left[1-e^{-\delta x_{(i)}^{2}}\right]$$
$$-2\sum_{i=1}^{k}\ln\left[1+\left[1-e^{-\delta x_{(i)}^{2}}\right]^{\lambda}\right] + (n-k)\left[\ln\left(1-\left[1-e^{-\delta T^{2}}\right]^{\lambda}\right) - \ln\left(1+\left[1-e^{-\delta T^{2}}\right]^{\lambda}\right)\right]. \tag{18}$$

For simplicity write $\mathrm{N}_i = \left[1-e^{-\delta x_{(i)}^{2}}\right]$ and $\mathrm{N}_T = \left[1-e^{-\delta T^{2}}\right]$, then obtaining the partial derivatives of (18) as follows

$$\frac{\partial \ln l_1}{\partial\lambda} = \frac{k}{\lambda} + \sum_{i=1}^{k}\ln N_i - 2\sum_{i=1}^{k}\frac{\ln N_i}{\left[\left[N_i\right]^{-\lambda}+1\right]} - (n-k)\left[\frac{\ln N_T}{\left[\left[N_T\right]^{-\lambda}-1\right]} + \frac{\ln N_T}{\left[\left[N_T\right]^{-\lambda}+1\right]}\right],$$

and

$$\frac{\partial \ln l_1}{\partial\delta} = \frac{k}{\delta} - \sum_{i=1}^{k}x_{(i)}^{2} + (\lambda-1)\sum_{i=1}^{k}\frac{x_{(i)}^{2}e^{-\delta x_i^{2}}}{N_i} - 2\lambda\sum_{i=1}^{k}\frac{x_{(i)}^{2}e^{-\delta x_i^{2}}[N_i]^{\lambda-1}}{1+\left[N_i\right]^{\lambda}}$$
$$-(n-k)\left[\sum_{i=1}^{n}\frac{T^{2}\lambda e^{-\delta T^{2}}[N_T]^{\lambda-1}}{1-\left[N_T\right]^{\lambda}} + \sum_{i=1}^{n}\frac{T^{2}\lambda e^{-\delta T^{2}}[N_T]^{\lambda-1}}{1+\left[N_T\right]^{\lambda}}\right],$$

and equating these partial derivatives to zero and solving simultaneously yield the MLE's of λ and δ based on Type I censored samples.

5.2 Maximum likelihood estimators in case of type-II censored samples

Consider $X_{(1)} < X_{(2)} <,\ldots,< X_{(k)}$ be a Type-II censoring sample of size n observed from lifetime testing experiment whose lifetime have the density function (5). The log-likelihood based on Type II censoring is given by:

$$\ln l_2 = \ln\left(\frac{n!}{(n-k)!}\right) + k\ln 2\lambda + k\ln\delta + k\ln 2 + \sum_{i=1}^{k}\ln\left(x_{(i)}\right) - \delta\sum_{i=1}^{k}x_{(i)}^{2} + (\lambda-1)\sum_{i=1}^{k}\ln\left[1-e^{-\delta x_{(i)}^{2}}\right]$$
$$-2\sum_{i=1}^{k}\ln\left[1+\left[1-e^{-\delta x_{(i)}^{2}}\right]^{\lambda}\right] + (n-k)\left[\ln\left(1-\left[1-e^{-\delta x_{(k)}^{2}}\right]^{\lambda}\right) - \ln\left(1+\left[1-e^{-\delta x_{(k)}^{2}}\right]^{\lambda}\right)\right]. \tag{19}$$

For simplicity write $\mathrm{N}_i = \left[1-e^{-\delta x_{(i)}^{2}}\right]$ and $\mathrm{N}_k = \left[1-e^{-\delta x_{(k)}^{2}}\right]$, then obtaining the partial derivatives of (19) as follows

$$\frac{\partial \ln l_2}{\partial\lambda} = \frac{k}{\lambda} + \sum_{i=1}^{k}\ln N_i - 2\sum_{i=1}^{k}\frac{\ln N_i}{\left[\left[N_i\right]^{-\lambda}+1\right]} - (n-k)\left[\frac{\ln N_k}{\left[\left[N_k\right]^{-\lambda}-1\right]} + \frac{\ln N_k}{\left[\left[N_k\right]^{-\lambda}+1\right]}\right],$$

and

$$\frac{\partial \ln l_2}{\partial \delta} = \frac{k}{\delta} - \sum_{i=1}^{k} x_{(i)}{}^2 + (\lambda-1)\sum_{i=1}^{k} \frac{x_{(i)}{}^2 e^{-\delta x_{(i)}{}^2}}{N_i} - 2\lambda \sum_{i=1}^{k} \frac{x_{(i)}{}^2 e^{-\delta x_{(i)}{}^2} [N_i]^{\lambda-1}}{1+[N_i]^{\lambda}}$$
$$-(n-k)\left[\sum_{i=1}^{n} \frac{\lambda x_{(k)}{}^2 e^{-\delta T^2} [N_k]^{\lambda-1}}{1-[N_k]^{\lambda}} + \sum_{i=1}^{n} \frac{\lambda x_{(k)}{}^2 e^{-\delta x_{(k)}{}^2} [N_k]^{\lambda-1}}{1+[N_k]^{\lambda}}\right].$$

Setting $\partial \ln l_2 / \partial \lambda$ and $\partial \ln l_2 / \partial \delta$ equal to zero and solving these equations numerically yield the MLE's of λ and δ based on Type II censored sample.

5.3 Numerical studies

In this subsection, we provide a numerical study to evaluate the performance of the maximum likelihood estimates of the TIIHLR based on Type I and Type II censoring schemes. The algorithm used here is designed as follows:

i. A random sample $X_{(1)} < X_{(2)} < \ldots < X_{(k)}$ of sizes n = 30, 50, 100, 200 and 300 are generated from the TIIHLR distribution under Type I and Type II censored samples.
ii. Select initial value for parameters as λ =0.5 and δ =0.25.
iii. Two termination times are selected as T=5 and 8.
iv. The number of failure items; k is selected, based on two censoring levels as 50% and 70%.
v. For each sample sizes, the estimates are obtained.
vi. For each sample, the experiment is repeated 10000 times, and MLEs of the parameters, their biases and MSEs are recorded.
vii. The simulation results are provided in Tables (3) and (4).

Table 3. MLEs, Biases, MSEs of TIIHLR distribution under Type I censored Samples

		δ =0.5, λ =0.5			δ =0.25 λ =0.5		
n	***T***	**MLE**	**Bias**	**MSE**	**MLE**	**Bias**	**MSE**
	5	0.5024	0.0024	0.0018	0.2523	0.0023	0.0005
50	8	0.5014	0.0014	0.0016	0.2512	0.0012	0.0004
	5	0.5150	0.0150	0.0089	0.5162	0.0162	0.0087
	8	0.5147	0.0147	0.0085	0.5150	0.0150	0.0082
	5	0.5000	0.0000	0.0008	0.2498	-0.0002	0.0002
100	8	0.5023	0.0023	0.0008	0.2498	-0.0002	0.0002
	5	0.5094	0.0094	0.0041	0.5110	0.0110	0.0041
	8	0.5047	0.0047	0.0037	0.5126	0.0126	0.0041
	5	0.5015	0.0015	0.0005	0.2504	0.0004	0.0001
200	8	0.5005	0.0005	0.0004	0.2502	0.0002	0.0001
	5	0.5032	0.0032	0.0018	0.5029	0.0029	0.0018
	8	0.5041	0.0041	0.0020	0.5062	0.0062	0.0019
	5	0.5012	0.0012	0.0003	0.2494	-0.0006	0.0001
300	8	0.5013	0.0013	0.0003	0.2507	0.0007	0.0001
	5	0.5017	0.0017	0.0012	0.5023	0.0023	0.0012
	8	0.5018	0.0018	0.0011	0.5013	0.0013	0.0013

Table 4. MLEs, Biases, MSEs of TIIHLR distribution under Type II censored Samples

			δ =0.5, λ =0.5			δ =0.25, λ =0.5	
n	$X_{(k)}$	**MLE**	**Bias**	**MSE**	**MLE**	**Bias**	**MSE**
	50%	0.3774	-0.1226	0.0175	0.1877	-0.0623	0.0045
50	80%	0.4158	-0.0842	0.0088	0.2073	-0.0427	0.0023
	50%	0.5766	0.0766	0.0940	0.5774	0.0774	0.0837
	80%	0.5249	0.0249	0.0158	0.5263	0.0263	0.0157
	50%	0.3734	-0.1266	0.0171	0.1863	-0.0637	0.0043
100	80%	0.4134	-0.0866	0.0084	0.2059	-0.0441	0.0021
	50%	0.5315	0.0315	0.0225	0.5407	0.0407	0.0265
	80%	0.5101	0.0101	0.0065	0.5114	0.0114	0.0060
	50%	0.3715	-0.1285	0.0171	0.1858	-0.0642	0.0043
200	80%	0.4112	-0.0888	0.0083	0.2053	-0.0447	0.0021
	50%	0.5153	0.0153	0.0080	0.5144	0.0144	0.0097
	80%	0.5058	0.0058	0.0028	0.5062	0.0062	0.0033
	50%	0.3708	-0.1292	0.0171	0.1857	-0.0643	0.0042
300	80%	0.4102	-0.0898	0.0083	0.2054	-0.0446	0.0021
	50%	0.5104	0.0104	0.0053	0.5104	0.0104	0.0056
	80%	0.5062	0.0062	0.0020	0.5040	0.0040	0.0021

From Table 3 we conclude that; as the sample size *n* increases the MSE of estimates decreases. Also, as the termination time *T* increases, the MSE of estimates decreases. Based on Table 4, we can see that as the sample size *n* increases the MSE of estimates decreases. Also, as the censoring level time $X_{(k)}$ increases, the MSE of estimates decreases.

6 Applications

To illustrate the importance and flexibility of the TIIHLR distribution, two real data sets are demonstrated. We compare the fits of the TIIHLR model with some models namely; the *transmuted Rayleigh* (TR) (see Merovci [25], *transmuted generalized Rayleigh* (TGR) (see Merovci [26], *exponentiated transmuted generalized Rayleigh* (ETGR) (see Ahmed et al. [27]) and *transmuted exponentiated inverse Rayleigh distribution* (TEIR) (see Haq [28]) distributions. Their associated densities are given, respectively, by

$$f_{TR}(x)=\frac{x}{\beta^2}e^{-\left(\frac{x^2}{2\beta^2}\right)}\left(1+\lambda-2\lambda e^{-\left(\frac{x^2}{2\beta^2}\right)}\right),$$

$$f_{TGR}(x)=2\alpha\beta^2 x\, e^{(\beta x)^2}\left(1-\exp(\beta x)^2\right)^{\alpha-1}\left(1+\lambda-2\lambda\left(1-\exp(\beta x)^2\right)^{\alpha}\right),$$

$$f_{ETGR}(x)=2\alpha\delta\beta^2 x\, \mathrm{e}^{-(\beta x)^2}\left(1-\mathrm{e}^{-(\beta x)^2}\right)^{\delta\alpha-1}\left(1+\lambda-2\lambda\left(1-\mathrm{e}^{-(\beta x)^2}\right)^{\alpha}\right)\left(1+\lambda-\lambda\left(1-\mathrm{e}^{-(\beta x)^2}\right)^{\alpha}\right)^{\delta-1},$$

$$f_{TEIR}(x)=\frac{2\alpha\theta}{x^3}\exp\left[-\left(\frac{\theta\alpha}{x^2}\right)\right]\left(1+\lambda-2\lambda\exp\left[-\left(\frac{\theta\alpha}{x^2}\right)\right]\right).$$

Criteria like; the *maximised log-likelihood* (-2ℓ), *Akaike information criterion* (AIC), the, *Bayesian information criterion* (BIC), *Anderson-Darling* (A^*) and *Cramér-von Mises* (W^*) statistics are selected.

6.1 First real data set

The first data set (gauge lengths of 10 mm) obtained from Kundu and Raqab [29]. This data set consists of, 63 observations are listed as follows: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

MLEs of the model parameters are provided in Table (5). While Table (6) summarises the values of AIC, BIC, (-2ℓ), A^* and W^*.

Table 5. Estimated parameters for first data set

Models	Estimated Parameters			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\delta}$
TIIHLR *(λ, δ)*			11.244	0.268389
TR *(β, λ)*		1.82396	1	
TGR*(α, β, λ)*	6.2143	0.5021	0.1207	
TEIR *(α, β, λ,)*	1.20545	10.1136	1	
ETGR*(α, β, λ, δ)*	0.169106	0.530147	0.89804	414.148

Table 6. Goodness measures for estimates for the first data set

Models	AIC	BIC	-2ℓ	A^*	W^*
TIIHLR *(λ, δ)*	117.034	121.320	-56.517	0.36735	0.069889
TR *(β, λ)*	155.572	159.858	-75.7858	5.4111	0.906164
TGR*(α, β, λ)*	122.640	126.926	-59.320	0.38047	0.084888
TEIR *(α, β, λ,)*	157.009	163.438	-75.5045	5.49173	0.920139
ETGR*(α, β, λ, δ)*	122.975	131.573	-57.1295	0.38234	0.077132

We reveal from Table (6) that the value of criteria's of the TIIHLR distribution is smaller than the criteria's values of the competitive models. So, the TIIHLR distribution seems to be a very competitive model to these data. Further, the plots of the histogram and empirical cdf of the first data with the estimated densities obtained using maximum likelihood procedure are represented in Fig. 3. We also observe from this figure that the TIIHLR distribution provides an adequate fit to the data than the other competing models.

6.2 Second real data set

The second data set (gauge lengths of 20 mm) is also obtained from Kundu and Raqab [29]. These data set consists of 74 observations and are listed as follows: 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

MLEs of the model parameters are provided in Table (7). Whereas Table (8) contains the values of AIC, BIC, -2ℓ, $A^*$and W^*

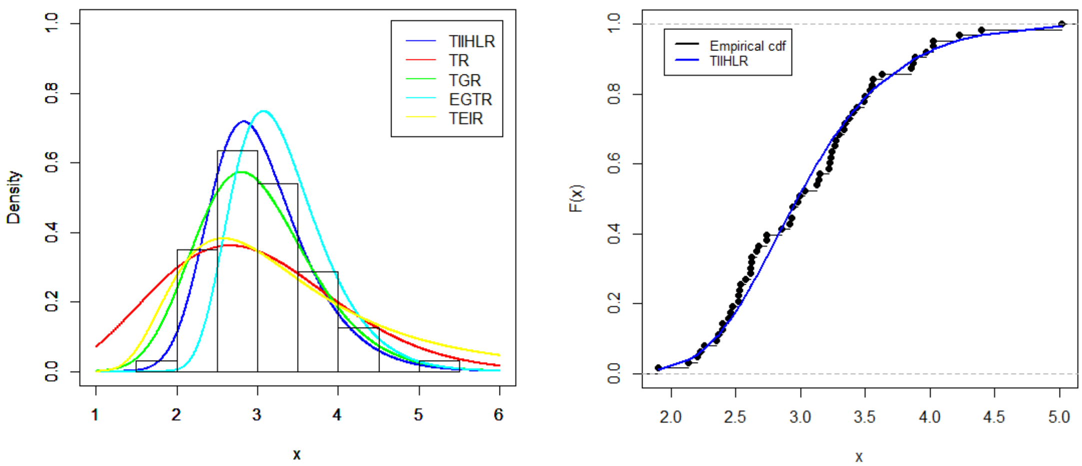


Fig. 3. Estimated pdf and cdf plots of models for the first data set

Table 7. Estimated parameters for the second data set

Models	**Estimated Parameters**			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\delta}$
TIIHLR *(λ, δ)*			8.58862	0.359385
TR *(β, λ)*		1.47664	1	
TGR*(α, β, λ)*	7.57895	0.692186	-0.635265	
TEIR *(α, β, λ,)*	1.33763	5.87032	1	
ETGR*(α, β, λ, δ)*	2.1214	0.6985	0.3201	7.790

Table 8. Goodness measures for estimates for the second data set

Models	**AIC**	**BIC**	-2ℓ	A^*	W^*
TIIHLR *(λ, δ)*	115.095	119.703	-55.5474	0.913365	0.132946
TR *(β, λ)*	150.965	155.573	-73.4826	6.72007	1.18901
TGR*(α, β, λ)*	127.61	132.218	-61.8050	8.1249	1.71242
TEIR *(α, β, λ,)*	157.29	164.202	-75.6448	6.27413	1.08763
ETGR*(α, β, λ, δ)*	121.400	130.616	-56.7121	1.13023	0.164683

We detect from the values of criteria for the TIIHLR model in Table 8 are smaller than the values of corresponding competitive models. So, the TIIHLR distribution will be chosen as the best model for the data. Also, the plots of the histogram and empirical cdf of the second data are shown in Fig. 4. We also observe from this figure that the TIIHLR distribution provides an adequate fit to the data than the other competing models.

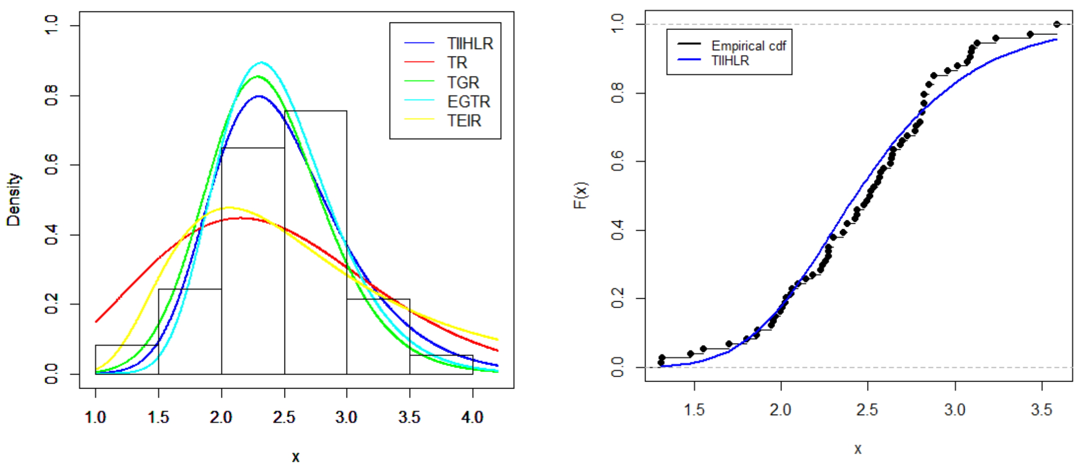


Fig. 4. Estimated pdf and cdf plots of models for the second data set

7 Conclusion

In this paper, we propose a two-parameter model, named the TIIHLR distribution. The TIIHLR model is motivated by the wide use of the Rayleigh distribution in practice and also for the fact that the generalisation provides more flexibility to analyse positive real-life data. We derive explicit expressions for the ordinary and incomplete moments, order statistics, and Rényi entropy. The maximum likelihood estimators of the model parameters are investigated based on complete and censored samples. We provide some simulation results to assess the performance of the proposed model. An application to real life data shows that the TIIHLR distribution is a strong and better competitor than the transmuted Rayleigh, transmuted generalised Rayleigh, exponentiated transmuted generalised Rayleigh and transmuted exponentiated inverse Rayleigh distributions.

Competing Interests

Authors have declared that no competing interests exist.

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