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New Extension of Kannan and Chatterjea Fixed Point Theorems on Complete Metric Spaces

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Authors' contributions

This work was carried out in collaboration between all authors. Author RM designed the study, wrote the protocol and supervised the work. Author AM wrote the first draft of the manuscript. Authors SM and KA produced the examples which actually present the justification of the obtained results. Authors RM and KA managed the literature searches and edited the manuscript. All authors read and approved the final manuscript.

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Abstract

The theory of fixed point is applied in many fields of mathematics as well as in other studies. That is the main reason for producing new results, as well as for generalizing already known theorems about fixed point. This paper considers several generalizations of R. Kannan [1] and S. K. Chatterjea [2] theorems about fixed point.

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1 Introduction

The theory of fixed point dynamically develops in the period of the recent decades. The first important result in the theory of fixed point about contractive mapping is Banach theorem (principle of contractive mapping, [3]). Exactly this theorem is very important researching instrument in many different fields of mathematics. The above theorem was presented, 1922, in S. Banach dissertation. By applying the stated theorem is proven that an integral equality might be solved. Further, R. Kannan [1] 1968 has proven that, if (X,d) is a complete metric space and $S:X\to X$ is such mapping that it exists $\alpha\in(0,\frac{1}{2})$ so that for all $x,y\in X$ the inequality

$$d(Sx,Sy)\le\alpha(d(x,Sx)+d(y,Sy)), \tag{1}$$

is satisfied, then there is a unique fixed point on S. Several years later, 1972, S. K. Chatterjea [2] has proven that if (X,d) is a complete metric space and $S:X\to X$ is such mapping that it exists $\alpha\in(0,\frac{1}{2})$ so that for all $x,y\in X$ the inequality

$$d(Sx,Sy)\le\alpha(d(x,Sy)+d(y,Sx)), \tag{2}$$

is satisfied, then there is a unique fixed point on S.

In the further considerations we will generalize these results by applying the sequentially convergent mappings, defined as the following

Definition 1 [4]. Let (X,d) be a metric space. A mapping $T:X\to X$ is said sequentially convergent if, for each sequence $\{y_n\}$ the following holds true:

if $\{Ty_n\}$ convergences, then $\{y_n\}$ also convergences.

2 Generalization of Kannan Theorem

Theorem 1. Let (X,d) be a complete metric space, $S:X\to X$ and $T:X\to X$ be a mapping such that it is continuous, injection and sequentially convergent. If $\alpha>0$ $\gamma\ge0$, $2\alpha+\gamma<1$ and

$$d(TSx,TSy)\le\alpha(d(Tx,TSx)+d(Ty,TSy))+\gamma d(Tx,Ty) \tag{3}$$

for all $x,y\in X$, then there is a unique fixed point on S and for any $x_0\in X$ the sequence $\{S^n x_0\}$ convergences to the above fixed point.

Proof. Let x_0 be any point on X and the sequence $\{x_n\}$ be defined as the following $x_{n+1}=Sx_n$, $n=0,1,2,3,...$. The inequality (3) implies that

$$d(Tx_{n+1},Tx_n)\le\alpha(d(Tx_{n+1},Tx_n)+d(Tx_n,Tx_{n-1}))+\gamma d(Tx_n,Tx_{n-1}),$$

therefore

$$d(Tx_{n+1},Tx_n)\le\lambda d(Tx_n,Tx_{n-1}), \tag{4}$$

for each $n = 0,1,2,3,...$, and $0 < \lambda = \frac{\alpha+\gamma}{1-\alpha} < 1$. The inequality (4) implies that for all $m,n \in \mathbf{N}$, $n > m$

$$d(Tx_n, Tx_m) \le \frac{\lambda^m}{1-\lambda} d(Tx_1, Tx_0) \quad (5)$$

holds true. And since $0 < \lambda < 1$, it is true that the sequence $\{Tx_n\}$ is Caushy. But, X is complete metric space, thereby (5) implies that the sequence $\{Tx_n\}$ is convergent, i.e. it exists $z \in X$ so that $\lim_{n\to\infty} Tx_n = z$. Further, the mapping $T : X \to X$ is sequentially convergent and since the sequence $\{Tx_n\}$ is convergent, it implies that the sequence $\{x_n\}$ is also convergent, i.e. it exists $u \in X$ so that $\lim_{n\to\infty} x_n = u$ holds true. Further, the continuous of T implies that $\lim_{n\to\infty} Tx_n = Tu$ holds true. Thus,

$$\begin{aligned} d(TSu,Tu) &\le d(TSu,Tx_n) + d(Tx_n,Tx_{n+1}) + d(Tx_{n+1},Tu) \\ &= d(TSu,TS^n x_0) + d(Tx_n,Tx_{n+1}) + d(Tx_{n+1},Tu) \\ &\le \alpha[d(Tu,TSu) + d(TS^{n-1}x_0,TS^n x_0)] + \gamma d(Tu,TS^{n-1}x_0) + \lambda^n d(Tx_1,Tx_0) + d(Tx_{n+1},Tu) \\ &= \alpha d(Tu,TSu) + \alpha\lambda^{n-1} d(Tx_1,Tx_0) + \gamma d(Tu,Tx_{n-1}) + \lambda^n d(Tx_1,Tx_0) + d(Tx_{n+1},Tu). \end{aligned}$$

For $n \to \infty$, the continuous of T and $0 < \lambda < 1$ imply that $d(TSu,Tu) \le \alpha d(TSu,Tu)$. But, since $0 < \alpha < 1$, the latter implies that $d(TSu,Tu) = 0$, i.e. $TSu = Tu$. Finally, T is injection, and thus $Su = u$. The latter actually means that the mapping S has a fixed point.

Let $u,v \in X$ be fixed points on S, i.e. $Su = u$ and $Sv = v$. Then, (3) implies the following

$$d(Tu,Tv) = d(TSu,TSv) \le \alpha(d(Tu,TSu) + d(Tv,TSv)) + \gamma d(Tu,Tv) = \gamma d(Tu,Tv)$$

Since $0 \le \gamma < 1$, the last inequality implies that $d(Tu,Tv) = 0$, i.e. $Tu = Tv$. But, since T is injection, it is true that $u = v$. The last actually means that T has a unique fixed point. Finally, the arbitrariness of $x_0 \in X$ and the above stated imply that for each $x_0 \in X$ the sequence $\{S^n x_0\}$ convergences to the unique fixed point on S. ■

Consequence 1. Let (X,d) be a complete metric space, $S : X \to X$ and $T : X \to X$ be such mapping that it is continuous, injection and sequentially convergent. If $\lambda \in (0,1)$ and

$$d(TSx,TSy) \le \lambda \sqrt[3]{d(Tx,TSx) \cdot d(Ty,TSy) \cdot d(Tx,Ty)}$$

holds true for all $x,y \in X$, then S has a unique fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ convergences to the above fixed point.

Proof. The inequality of arithmetic and geometric means (the AM-GM inequality) implies that

$$d(TSx,TSy) \le \frac{\lambda}{3}(d(Tx,TSx) + d(Ty,TSy) + d(Tx,Ty)) .$$

The statement is directly implied by theorem 1 for $\alpha = \gamma = \frac{\lambda}{3}$. ■

Consequence 2. Let (X,d) be a complete metric space, $S:X\to X$ and the mapping $T:X\to X$ be continuous, injection and sequentially convergent. If $\alpha>0$, $\gamma\geq 0$, $2\alpha+\gamma<1$ and

$$d(TSx,TSy)\leq \alpha\frac{d^2(Tx,TSx)+d^2(Ty,TSy)}{d(Tx,TSx)+d(Ty,TSy)}+\gamma d(Tx,Ty),$$

holds true for all $x,y\in X$, then S has a fixed point and for each $x_0\in X$ the sequence $\{S^n x_0\}$ convergences to a fixed point.

Proof. The inequality stated in the condition implies the inequality (3). And the claim is implied by the Theorem 1. ■

Consequence 3 [5]. Let (X,d) be a complete metric space, $S:X\to X$ and the mapping $T:X\to X$ be continuous, injection and sequentially convergent. If $\alpha\in(0,\frac{1}{2})$ and

$$d(TSx,TSy)\leq \alpha(d(Tx,TSx)+d(Ty,TSy)) \tag{6}$$

for all $x,y\in X$, then S has a unique fixed point and for each $x_0\in X$ the sequence $\{S^n x_0\}$ convergences exactly to that fixed point.

Proof. It is sufficient to take $\gamma=0$ in the Theorem 1. ■

The paper [5] considers an example in which Kannan Theorem is not applicable, but the Consequence 3 (Theorem 2.1. [5]) implies the existence of a fixed point for the considered mapping. Hereinafter we will consider one more example of this type.

Example 1. Let $X=\{0\}\cup\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},...\}$ and d be an Euclidian metric on X. Then, (X,d) is a complete metric space. Let the mapping $S:X\to X$ be determined as the following: $S(0)=0$ and $S(\frac{1}{n})=\frac{1}{n+1}$, for $n\geq 1$. If there exists $\alpha\in(0,\frac{1}{2})$, so that for all $x,y\in X$ the condition (1) is satisfied, and for $x=\frac{1}{n-1}$, $y=\frac{1}{2n-1}$ then for each $n>1$ the following should be satisfied $2<\frac{1}{\alpha}\leq\frac{5n-3}{(n-1)(2n-1)}$, which is contradictory. Therefore the Kannan Theorem is not applicable when solving the given problem. The mapping $T:X\to X$ determined as $T(0)=0$ and $T(\frac{1}{n})=\frac{1}{[e^{2n}]}$, for $n\geq 1$ is continuous, injection and sequentially convergent. Further, since $[x]\cdot[y]\leq[xy]$, for all $x,y\geq 0$ it is true that for each $n\geq 1$.

$$7[e^{2n}]\leq[e^2]\cdot[e^{2n}]=[e^2e^{2n}]=[e^{2(n+1)}],\text{ i.e. } \frac{1}{[e^{2(n+1)}]}\leq\frac{1}{6}\left(\frac{1}{[e^{2n}]}-\frac{1}{[e^{2(n+1)}]}\right)$$

is satisfied. Therefore, for all $m,n\in\mathbf{N}$, $m>n$

$$\begin{aligned}|TS(\tfrac{1}{n})-TS(\tfrac{1}{m})|&=|\frac{1}{[e^{2(n+1)}]}-\frac{1}{[e^{2(m+1)}]}|<\frac{1}{[e^{2(n+1)}]}\leq\frac{1}{6}\left(\frac{1}{[e^{2n}]}-\frac{1}{[e^{2(n+1)}]}\right)\\&\leq\frac{1}{6}\left[\frac{1}{[e^{2n}]}-\frac{1}{[e^{2(n+1)}]}+\frac{1}{[e^{2m}]}-\frac{1}{[e^{2(m+1)}]}\right]\\&\leq\frac{1}{6}[|T(\tfrac{1}{n})-TS(\tfrac{1}{n})|+|T(\tfrac{1}{m})-TS(\tfrac{1}{m})|]\end{aligned}$$

holds true, and for each $n \in \mathbf{N}$

$$|TS(0) - TS(\tfrac{1}{n})| = \frac{1}{[e^{2(n+1)}]} \le \frac{1}{6}(\frac{1}{[e^{2n}]} - \frac{1}{[e^{2(n+1)}]}) = \frac{1}{6}[|T(0) - TS(0)| + |T(\tfrac{1}{n}) - TS(\tfrac{1}{n})|]$$

also holds true. Thus, the inequality (6) holds true for $\alpha = \frac{1}{6}$. Therefore, the Consequence 3 implies that the mapping S has a unique fixed point. ■

Consequence 4 [1]. If (X,d) is a complete metric space and $S: X \to X$ is such mapping that it exists $\alpha \in (0,\frac{1}{2})$ so that for all $x, y \in X$ the condition (1) is satisfied, then S has a unique fixed point.

Proof. It is sufficient $Tx = x$ to be substitute in Consequence 3. The mapping T is continuous, injection and sequentially convergent, and furthermore the condition (1) is equivalent to the condition (6). ■

Remark 1. For $Tx = x$, the Theorem 1 and the consequences 1 and 2, according to the same arguments as the proof of Consequence 4, imply that into a complete metric space X, a mapping $S: X \to X$ such that it satisfies one of the following conditions

$$d(Sx,Sy) \le \alpha[d(x,Sx) + d(y,Sy)] + \gamma d(x,y),\ \alpha > 0,\ \gamma \ge 0,\ 2\alpha + \gamma < 1,$$
$$d(Sx,Sy) \le \lambda\sqrt[3]{d(x,Sx)\cdot d(y,Sy)\cdot d(x,y)},\ \lambda \in (0,1),$$
$$d(Sx,Sy) \le \alpha\frac{d^2(x,Sx)+d^2(y,Sy)}{d(x,Sx)+d(y,Sy)} + \gamma d(x,y),\ \alpha > 0,\ \gamma \ge 0,\ 2\alpha + \gamma < 1,$$

has a unique fixed point $u \in X$, and exactly that fixed point is the bound of the sequence $\{x_n\}$ defined as $x_{n+1} = Sx_n$, for $n = 0,1,2,3,...$, and x_0 is any point on X.

3 Generalization of Chatterjea Theorem

Theorem 2. Let (X,d) be a complete metric space, $S: X \to X$ and $T: X \to X$ be continuous, injection and sequentially convergent mapping. If $\alpha > 0$, $\gamma \ge 0$, $2\alpha + \gamma < 1$ and

$$d(TSx,TSy) \le \alpha(d(Tx,TSy) + d(Ty,TSx)) + \gamma d(Tx,Ty) \tag{7}$$

for all $x, y \in X$, then S has a unique fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ convergences to that fixed point.

Proof. Let x_0 be any point on X and let the sequence $\{x_n\}$ be defined as $x_{n+1} = Sx_n$, $n = 0,1,2,3,...$. The inequality (7) implies the following

$$\begin{aligned} d(Tx_{n+1},Tx_n) = d(TSx_n,TSx_{n-1}) &\le \alpha(d(Tx_n,TSx_{n-1}) + d(Tx_{n-1},TSx_n)) + \gamma d(Tx_n,Tx_{n-1}) \\ &= \alpha d(Tx_{n+1},Tx_{n-1}) + \gamma d(Tx_n,Tx_{n-1}) \\ &\le \alpha(d(Tx_{n+1},Tx_n) + d(Tx_n,Tx_{n-1})) + \gamma d(Tx_n,Tx_{n-1}) \end{aligned}$$

therefore,

$$d(Tx_{n+1},Tx_n) \le \lambda d(Tx_{n-1},Tx_n). \tag{8}$$

for each $n=0,1,2,3,...$ and $\lambda=\frac{\alpha+\gamma}{1-\alpha}<1$. Thus, the inequality (8) implies that for all $m,n\in \mathbf{N}$, $n>m$ it is true that

$$d(Tx_n,Tx_m)\leq \frac{\lambda^m}{1-\lambda}d(Tx_1,Tx_0)\,, \tag{9}$$

Since $\lambda<1$, the latter implies that the sequence $\{Tx_n\}$ is Cauchy sequence. Further, analogously as the proof of Theorem 1 we conclude that the sequence $\{Tx_n\}$ is convergent, and further the sequence $\{x_n\}$ is convergent, i.e. it exists $u\in X$ so that $\lim\limits_{n\to\infty} x_n=u$ and $\lim\limits_{n\to\infty} Tx_n=Tu$. Thus,

$$\begin{aligned} d(TSu,Tu) &\leq d(TSu,TS^n x_0)+d(TS^n x_0,TS^{n+1}x_0)+d(TS^{n+1}x_0,Tu)\\ &\leq \alpha(d(Tu,TS^n x_0)+d(TS^{n-1}x_0,TSu))+\gamma d(Tu,TS^{n-1}x_0)+\lambda^n d(Tx_1,Tx_0)+d(Tx_{n+1},Tu)\\ &=\alpha d(Tu,Tx_n)+\alpha d(Tx_{n-1},TSu)+\gamma d(Tu,Tx_{n-1})+\lambda^n d(Tx_1,Tx_0)+d(Tx_{n+1},Tu). \end{aligned}$$

Since $0<\lambda<1$ and furthermore since T is continuous mapping, for $n\to\infty$ it implies that $d(TSu,Tu)\leq\alpha d(TSu,Tu)$. But, $\alpha<1$, therefore the latter implies that $d(TSu,Tu)=0$, i.e. $TSu=Tu$. Finally, T is injection, and therefore, $Su=u$, that is the mapping S has a unique fixed point.

Let $u,v\in X$ be fixed points on S, i.e. $Su=u$ and $Sv=v$. Thus, (7) implies that

$$d(Tu,Tv)=d(TSu,TSv)\leq\alpha(d(Tu,TSv)+d(Tv,TSu))+\gamma d(Tu,Tv)=(2\alpha+\gamma)d(Tu,Tv)$$

and since $2\alpha+\gamma<1$, the latter inequality implies that $d(Tu,Tv)=0$, i.e. $Tu=Tv$. But, T is injection, therefore $u=v$. The latter actually means that T has a unique fixed point and for each $x_0\in X$ the sequence $\{S^n x_0\}$ convergences to the unique fixed point on S. ■

Consequence 5. Let (X,d) be a complete metric space, $S:X\to X$ and $T:X\to X$ be continuous, injection and sequentially convergent. If $\lambda\in(0,1)$ and

$$d(TSx,TSy)\leq\lambda\sqrt[3]{d(Tx,TSy)\cdot d(Ty,TSx)\cdot d(Tx,Ty)}$$

for all $x,y\in X$, then S has a unique fixed point and furthermore for each $x_0\in X$ the sequence $\{S^n x_0\}$ convergences to exactly the above fixed point.

Proof. The arithmetic mean-geometric mean inequality implies that

$$d(TSx,TSy)\leq\frac{\lambda}{3}(d(Tx,TSy)+d(Ty,TSx)+d(Tx,Ty))\,.$$

For $\alpha=\gamma=\frac{\lambda}{3}$, by applying the Theorem 2 we actually get the above statement. ■

Consequence 6. Let (X,d) be a complete metric space, $S:X\to X$ and $T:X\to X$ be continuous mapping such that it is injection and sequentially convergent. If $\alpha>0$, $\gamma\geq 0$, $2\alpha+\gamma<1$ and

$$d(TSx,TSy)\leq\alpha\frac{d^2(Tx,TSy)+d^2(Ty,TSx)}{d(Tx,TSy)+d(Ty,TSx)}+\beta d(Tx,Ty)\,,$$

for all $x, y \in X$, then S has a unique fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ convergences to the above fixed point.

Proof. The inequality stated in the above condition implies the inequality (7). Thus, the statement is implied by Theorem 1. ■

Consequence 7. Let (X,d) be a complete metric space, $S: X \to X$ and $T: X \to X$ be continuous mapping such that it is injection and sequentially convergent. If $\alpha \in (0, \frac{1}{2})$ and

$$d(TSx,TSy) \le \alpha(d(Tx,TSy) + d(Ty,TSx)) \tag{10}$$

for all $x, y \in X$, then S has a unique fixed point and for each $x_0 \in X$ the sequence $\{S^n x_0\}$ convergences to the above fixed point.

Proof. For $\gamma = 0$ in the Theorem 2 we get the proof of the above statement. ■

Example 2. Let (X,d) and $T, S: X \to X$ be the metric space and the mappings defined as in the Example 1. If it exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the condition (2) is satisfied, then for $x = \frac{1}{n}$, $y = \frac{1}{2n}$ it is true that for each $n > 1$, $\frac{2n^2}{4n^2+3n+1} \le \alpha$ has to be satisfied. The latter is contradictory, thereby the sequence $\{\frac{2n^2}{4n^2+3n+1}\}$ convergences to $\frac{1}{2}$. Thus, the Chatterjea Theorem is not applicable for this case. But, it is easy to be proven that the mapping $T: X \to X$ satisfies the condition (10). Thus, the Consequence 7 implies that the mapping S has a unique fixed point. ■

Consequence 8 [2]. Let (X,d) be a complete metric space and $S: X \to X$ be such mapping that it exists $\alpha \in (0, \frac{1}{2})$ and for all $x, y \in X$ the condition (2) is satisfied. Then S has a unique fixed point.

Proof. Let $Tx = x$. Thus the Consequence 7 implies the validity of the above statement. ■

Remark 2. Let $Tx = x$. By applying the same arguments as in the proof of Consequence 4, the Theorem 2 and the consequences 5 and 6 imples that in a complete metric space X, any mapping $S: X \to X$ such that it satisfies one of the following conditions

$$d(Sx,Sy) \le \alpha[d(x,Sy) + d(y,Sx)] + \gamma d(x,y),\ \alpha > 0,\ \gamma \ge 0,\ 2\alpha + \gamma < 1,$$
$$d(Sx,Sy) \le \lambda \sqrt[3]{d(x,Sy) \cdot d(y,Sx) \cdot d(x,y)},\ \lambda \in (0,1),$$
$$d(Sx,Sy) \le \alpha \frac{d^2(x,Sy) + d^2(y,Sx)}{d(x,Sy) + d(y,Sx)} + \beta d(x,y),\ \alpha > 0, \beta \ge 0,\ 2\alpha + \beta < 1$$

has a unique fixed point $u \in X$, which is also a bound of the sequence $\{x_n\}$ defined as the following $x_{n+1} = Sx_n$, for $n = 0,1,2,3,\ldots$, and x_0 any point on X.

4 Generalization of Koparde – Waghmode Theorem

P. V. Koparde and B. B. Waghmode [6] have proven that, if (X,d) is a complete metric space and $S: X \to X$ is mapping such that it exists $\alpha \in (0, \frac{1}{2})$ so that for all $x, y \in X$ the following is satisfied

$$d^2(Sx,Sy) \le \alpha(d^2(x,Sx)+d^2(y,Sy)), \tag{11}$$

then S has a unique fixed point. In our further considerations, by applying sequentially convergent mappings, we will generalize this result.

Theorem 3. Let (X,d) be a complete metric space, $S:X\to X$ and $T:X\to X$ be a continuous mapping such that it is injection and sequentially convergent. If $\alpha>0$, $\gamma\ge 0$, $2\alpha+\gamma<1$ and

$$d^2(TSx,TSy) \le \alpha(d^2(Tx,TSx)+d^2(Ty,TSy))+\gamma d^2(Tx,Ty) \tag{12}$$

for all $x,y\in X$, then S has a unique fixed point and for each $x_0\in X$ the sequence $\{S^n x_0\}$ convergences to the above fixed point.

Proof. Let x_0 be any point on X and let the sequence $\{x_n\}$ be defined as the following $x_{n+1}=Sx_n$, $n=0,1,2,3,...$. The inequality (12) implies the following

$$\begin{aligned} d^2(Tx_{n+1},Tx_n) &= d^2(TSx_n,TSx_{n-1}) \le \alpha(d^2(Tx_n,TSx_n)+d^2(Tx_{n-1},TSx_{n-1}))+\gamma d^2(Tx_n,Tx_{n-1}) \\ &= \alpha(d^2(Tx_n,Tx_{n+1})+d^2(Tx_{n-1},Tx_n))+\gamma d^2(Tx_n,Tx_{n-1}), \end{aligned}$$

for each $n=0,1,2,3,...$. The condition given the Theorem implies that $\lambda=\sqrt{\frac{\alpha+\gamma}{1-\alpha}}<1$. The latter implies validity of the following $d(Tx_{n+1},Tx_n)\le\lambda d(Tx_n,Tx_{n-1})$, for each $n=0,1,...$. Analogously, as the Proofs of the Theorems 1 and 2, firstly the sequence $\{Tx_n\}$ is Cauchy, and further it is a convergent sequence. Further, the mapping $T:X\to X$ is sequentially convergent and thereby the sequence $\{Tx_n\}$ is convergent, it is true the sequence $\{x_n\}$ is also convergent, i.e. it exists $u\in X$ so that $\lim\limits_{n\to\infty} x_n=u$ holds true. Furthermore, the continuous of T implies that $\lim\limits_{n\to\infty} Tx_n=Tu$. We will prove that u is a fixed point on T. This,

$$\begin{aligned} d(TSu,Tu) &\le d(Tu,Tx_{n+1})+d(Tx_{n+1},TSu)=d(Tu,Tx_{n+1})+d(TSx_n,TSu) \\ &\le d(Tu,Tx_{n+1})+\sqrt{\alpha(d^2(Tx_n,TSx_n)+d^2(Tu,TSu))+\gamma d^2(Tu,Tx_n)} \\ &= d(Tu,Tx_{n+1})+\sqrt{\alpha(d^2(Tx_n,Tx_{n+1})+d^2(Tu,TSu))+\gamma d^2(Tu,Tx_n)} \end{aligned}$$

for each $n\in\mathbf{N}$. For $n\to\infty$, the latter is transformed as the following $d(TSu,Tu)\le\sqrt{\alpha}d(TSu,Tu)$. But, $\sqrt{\alpha}<1$. Therefore, $d(TSu,Tu)=0$. Now, one more time as in the proof of Theorem 1 we get that u is a fixed point on S.

Let $u,v\in X$ be two fixed points on S, i.e. $Su=u$ and $Sv=v$. Then (4) implies the following

$$\begin{aligned} d^2(Tu,Tv) &= d^2(TSu,TSv) \le \alpha(d^2(Tu,TSu)+d^2(Tv,TSv))+\gamma d^2(Tu,Tv) \\ &= \alpha(d^2(Tu,Tu)+d^2(Tv,Tv))+\gamma d^2(Tu,Tv)=\gamma d^2(Tu,Tv) \end{aligned}$$

and since $0\le\gamma<1$ the latter inequality implies that $d(Tu,Tv)=0$, i.e. $Tu=Tv$. But, T is injection, and thus $u=v$. That actually means that T has a unique fixed point. Finally, the arbitrariness of $x_0\in X$ and also the above stated, imply that for each $x_0\in X$ the sequence $\{S^n x_0\}$ convergences to the unique fixed point on S. ■

Consequence 9. Let (X,d) be a complete metric space, $S:X\to X$ and $T:X\to X$ be continuous mapping such that it is injection and sequentially convergent. If $\alpha\in(0,\frac{1}{2})$ and

$$d^2(TSx,TSy)\le\alpha(d^2(Tx,TSx)+d^2(Ty,TSy)) \tag{13}$$

for all $x,y\in X$, then S has a unique fixed point and for each $x_0\in X$ the sequence $\{S^n x_0\}$ convergences to the above fixed point.

Proof. For $\gamma=0$, the Theorem 3 implies the validity of the above statement. ■

Consequence 10 [6]. If (X,d) is a complete metric space and $S:X\to X$ is such mapping that it exists $\alpha\in(0,\frac{1}{2})$ and for all $x,y\in X$ the condition (11) is satisfied, then S has a unique fixed point.

Proof. For $Tx=x$ in the Consequence 9 we get the proof of the above statement. ■

Example 3. Let (X,d) and $T,S:X\to X$ be the metric space and the mappings defined as in the Example 1. If it exists $\alpha\in(0,\frac{1}{2})$ such that for all $x,y\in X$ the condition (11) is satisfied, then for $x=\frac{1}{n-1}$, $y=\frac{1}{2n-1}$ it is true that for each $n>1$, the inequality $\frac{(n-1)^2(2n-1)^2}{4(2n-1)^2+(n-1)^2}\le\alpha<\frac{1}{2}$ has to be satisfied, which is contradictory. Thus, the Koparde and Waghmode Theorem is not applicable when solving this problem. The mapping T is continuous, injection and also sequentially convergent and for all $m,n\in\mathbf{N}$, $n>m$ it is true that

$$|TS(\tfrac{1}{n})-TS(\tfrac{1}{m})|\le\tfrac{1}{6}\sqrt{2(|T(\tfrac{1}{n})-TS(\tfrac{1}{n})|^2+|T(\tfrac{1}{m})-TS(\tfrac{1}{m})|^2)}\,,$$

i.e.

$$|TS(\tfrac{1}{n})-TS(\tfrac{1}{m})|^2\le\tfrac{1}{18}(|T(\tfrac{1}{n})-TS(\tfrac{1}{n})|^2+|T(\tfrac{1}{m})-TS(\tfrac{1}{m})|^2)\,.$$

Furthermore, for each $n\in\mathbf{N}$ the following holds true

$$\begin{aligned}|TS(0)-TS(\tfrac{1}{n})|^2&\le\tfrac{1}{36}(|T(0)-TS(0)|^2+|T(\tfrac{1}{n})-TS(\tfrac{1}{n})|^2)\\&<\tfrac{1}{18}(|T(0)-TS(0)|^2+|T(\tfrac{1}{n})-TS(\tfrac{1}{n})|^2).\end{aligned}$$

Thus, for $\alpha=\frac{1}{18}$ the inequality (13) is satisfied. So, the Consequence 9 implies that the mapping S has a unique fixed point. ■

Remark 3. Let $Tx=x$, in the Theorem 3. By applying the same arguments as in the proof of Consequence 4, it is true that in a complete metric space X, any mapping $S:X\to X$ such that it satisfies the following condition

$$d^2(Sx,Sy)\le\alpha(d^2(x,Sx)+d^2(y,Sy))+\gamma d^2(x,y)\,,\ \alpha>0,\ \gamma\ge0,\ \ 2\alpha+\gamma<1\,,$$

has a unique fixed point $u\in X$, which is actually the bound of the sequence $\{x_n\}$ defined as the following $x_{n+1}=Sx_n$, $n=0,1,2,3,\ldots$, for any point x_0 on X.

5 Conclusion

In the above consideration, by applying the sequentially convergent mappings we have proven several generalizations of already known theorems about fixed point in a complete metric space. Further, we proved that many already known theorems in the theory of fixed point are implied by the obtained results in this paper. It is naturally to wonder do any other similar generalizations hold true for the already known theorems about common fixed points on two different mappings in a complete metric space and is it true that analogously other results might be generalized, such the results proven in [7–10].

Competing Interests

Authors have declared that no competing interests exist.

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