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A Proposed Method for Numerical Integration

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Authors' contributions

This work was carried out in collaboration between all authors. Author FOM designed the study, wrote the protocol and supervised the work. Authors ENBQ and LA wrote the codes for the numerical computations and performed the statistical analysis. Author FOM managed the analyses of the study. Author FOM wrote the first draft of the manuscript. Author KAD managed the literature searches and performed preliminary manual computations. All authors read and approved the final manuscript.

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Abstract

The main objective of this paper is to propose a numerical integration method that provides improved estimates as compared to the Newton-Cotes methods of integration. The method is an extension of trapezoidal rule where after segmentation, the top part of each segment was further subdivided into rectangles and/or squares and triangles (approximate). The area of each segment is then obtained as the sum of areas of these geometric shapes and the area of the down part of the segments which is usually a rectangle. The process resulted in an improved formula for numerical integration which we derived in the paper. The proposed method was compared with some Newton-Cotes methods of integration and it outperformed. With the proposed method, one can provide estimates with predetermined desired absolute relative true errors.

Keywords: Numerical integration; Newton-Cotes methods; absolute relative true error.

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1 Introduction

As indicated by Kaw and Keteltas [1], integration is the process of measuring the area under a function plotted on a graph. The process also known as integral calculus has countless applications in a wide range of fields including engineering, statistics, finance, actuarial science, etc. [2-4,1,5] and biostatistics to estimate quantiles and various distribution functions. Numerical integration has increased greatly in estimating likelihoods and posterior distributions using Bayesian methods [6].

Sometimes, the evaluation of expressions involving these integrals can become very difficult, if not impossible. Due to this reason, a number of different numerical methods (numerical integration) have been developed to simplify the integral [7]. For example in statistics, emphasis in recent years on Bayesian and empirical Bayesian methods and on mixture models has greatly increased the importance of such numerical integration for computing likelihoods and posterior distributions and associated moments and derivatives [7].

Numerical integration involves the approximation of numerical values that cannot be integrated analytically [7]. It is sometimes referred to as quadrature which involves replacing the area under a curve by an area of a square. Several numerical integration methods such as Newton-Cotes, Romberg integration, Gauss Quadrature and Monte Carlo integration are used to evaluate those functions that can't be integrated analytically. Newton-Cotes methods use interpolating polynomials. Newton-Cotes methods such as the Trapezium rule, Simpson $1/3$ rule, Simpson $3/8$ rule and Boole's rule are special cases of 1st, 2nd, 3rd and 4th order polynomials used respectively and the Weddle's rule is also a special case of the 6^{th} order polynomial. The Trapezium rule and the numerical integration method we are proposing have no restriction on the number of segmentation. The number of segments for the Simpson $1/3$ rule must be even and for Simpson $3/8$ rule, the number of segments must be a multiple of 3. For the Boole's rule the number of segments must be a multiple of 4 and for Weddle's rule the number must be a multiple of 6.

The Newton-Cotes formula is a frequently used interpolator function in the form of a polynomial. This formula involves n points in the interval $[a,b]$ with $n-1$ order polynomial which passes through the abscissas x_i ($i = 0, 1, \dots, n$) equally spaced. Approximating the area under the curve $y = f(x)$ from $x = a$ to $x = b$, the closed Newton-Cotes formula employs Lagrange interpolating when fitting polynomials. Letting $x_0 = a, x_n = b \, and \, \Delta = \frac{(b-a)}{n}$, we have

$$f_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x). \tag{1}$$

where n in $f_n(x)$ stands for the order polynomial that approximates the function $y = f(x)$ given at $n+1$ data points as $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ and

$$L_i(x) = \prod_{k=1, i\neq k}^{n} \frac{x - x_k}{x_i - x_k}, \qquad i = 0,1,2, \dots, n \ . \tag{2}$$

where $L_i(x)$ is a weighting function that includes a product of $n - 1$ terms with terms of $i = k$ omitted [1].

Integrating $f(x)$ over $[a,b]$ and choosing $x_i = a + \frac{(b-a)i}{n}$ we have the Newton-Cotes rule;

$$A_{n+1}(f) = \int_a^b f(x)\, dx \approx \sum_{i=0}^{n} w_i f(x_i) \ . \tag{3}$$

where the weight w_i is determined by;

$$w_i = \int_a^b L_i(x)\,dx = \int_{x_0}^{x_1} \prod_{k=0,\,i\neq k}^{n} \frac{x - x_k}{x_i - x_k}\,dx\,,\, i = 0,1,2,\ldots,n. \tag{4}$$

When $n = 1,$ we have a simple trapezoidal rule of;

$$A_2 = \frac{\Delta}{2}\left[f(x_0) + f(x_1)\right]. \tag{5}$$

When $n = 2,$ it gives the Simpson $1/3$ rule of;

$$A_3 = \frac{\Delta}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]. \tag{6}$$

When $n = 3,$ it gives the Simpson $3/8$ rule of;

$$A_4 = \frac{3\Delta}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]. \tag{7}$$

When $n = 4$, it gives the Boole's rule of;

$$A_5 = \frac{2\Delta}{45}\left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right]. \tag{8}$$

When $n = 6$, it gives the Weddle's rule of;

$$A_6 = \frac{3\Delta}{10}\left[f(x_0) + 5f(x_1) + f(x_2) + 6f(x_3) + f(x_4) + 5f(x_5) + f(x_6)\right]. \tag{9}$$

Details of other numerical integration methods like Richardson's extrapolation, Romberg rule, Gauss quadrature rule, Euler method and so forth can be found in [8,9].

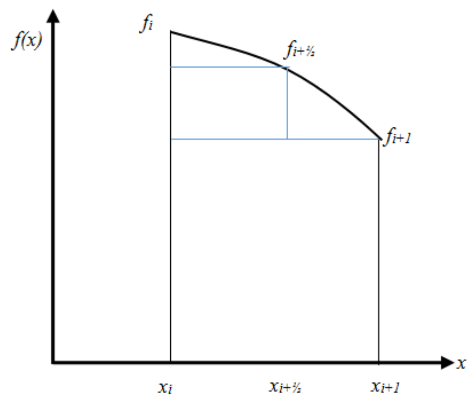
The main objective of this paper is to propose a numerical integration method that provides improved estimates as compared to the Newton-Cotes methods of integration.

2 Materials and Methods

2.1 Proposed numerical integration formula

Suppose the interval $[a,b]$ is subdivided into $n(n \in z^+)$ equal divisions each of width $\Delta = \frac{b-a}{n}$. Define x_i by $x_i = a + i\Delta, i = 0,1,2, \ldots, n$. Then $X_0 = a$ and $X_n = b$ Let $f_i = f(x_i), i = 0,1,2, \ldots, n$ be the ordinate at $x_i,\ i = 0,1,2, \ldots, n$ of the function f.

Suppose also that the interval $[x_i, x_{i+1}], i = 0, 1, 2, \ldots, n$ is divided into k equispaced points $x_i + \frac{t}{k}\Delta, t = 1, 2, \ldots, k$; then the corresponding ordinates of f are given by $f_{i+\frac{t}{k}} = f\left(x_i + \frac{t}{k}\Delta\right), t = 1, 2, \ldots, k;\ i = 0,1,2, \ldots, n-1$. Clearly when $t = k, x_i + \frac{k}{k}\Delta = x_{i+1}$ and $f_{i+\frac{k}{k}} = f_{i+1}$.



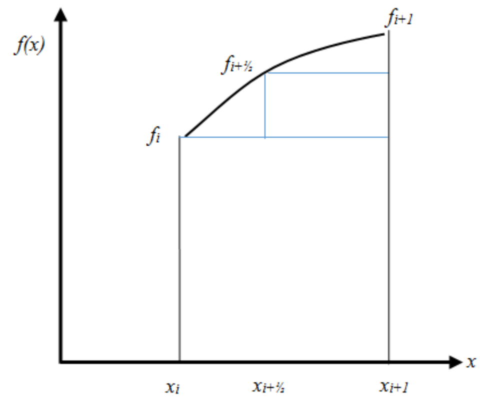


Fig. 1a. Monotone decreasing function of *f* **Fig. 1b. Monotone increasing function of *f***
Source: own research

As shown in Fig. 1a and Fig. 1b, the area under the curve for the i^{th} strip is estimated as

$$A_i^* = \frac{\Delta}{2k}\left[f_i + 2\sum_{t=1}^{k-1} f_{i+\frac{t}{k}} + f_{i+1} \right], i = 0,1,2,\ldots,n-1; k = 1,2,\ldots \tag{10}$$

when $k = 1$, we have

$$A_i^* = \frac{\Delta}{2}[\, f_i + f_{i+1}]. \tag{10a}$$

When $k = 2$, we have

$$A_i^* = \frac{\Delta}{4}\left[f_i + 2 f_{i+\frac{1}{2}} + f_{i+1} \right]. \tag{10b}$$

When $k = 3$, it becomes

$$A_i^* = \frac{\Delta}{6}\left[f_i + 2\left\{ f_{i+\frac{1}{3}} + f_{i+\frac{2}{3}} \right\} + f_{i+1} \right]. \tag{10c}$$

Here we show the relationship in (10) for $k = 2$. The relationship in (10) for $k \geq 3$ can similarly be obtained. We have left out the case of $k = 1$ because it is the same as the Trapezium Rule. Now, when

$k = 2$ an estimate of the area of the i^{th} strip under the curve is given by (the sum of areas of two triangles (approximate) and two rectangles as shown in the diagrams (Figs.1a and 1b) for *f* monotone increasing and decreasing.

$$A_i^* = \Delta \min(f_i, f_{i+1}) + \frac{\Delta}{2}\left[f_{i+\frac{1}{2}} - \min(f_i, f_{i+1}) \right] + \frac{\Delta}{4}\left[f_{i+\frac{1}{2}} - \min(f_i, f_{i+1}) \right] + \frac{\Delta}{4}\left[\max(f_i, f_{i+1}) - f_{i+\frac{1}{2}} \right] \tag{11}$$

If f is monotone increasing over (x_i, x_{i+1}), equation (11) reduces to

$$\begin{aligned} A_i^* &= \Delta f_i + \frac{\Delta}{2}\left[f_{i+\frac{1}{2}} - f_i \right] + \frac{\Delta}{4}\left[f_{i+\frac{1}{2}} - f_i \right] + \frac{\Delta}{4}\left[f_{i+1} - f_{i+\frac{1}{2}} \right] \\ &= \frac{\Delta}{4}\left[4 f_i - 2 f_{i+\frac{1}{2}} - 2 f_i + f_{i+\frac{1}{2}} - f_i + f_{i+1} - f_{i+\frac{1}{2}} \right] \\ &= \frac{\Delta}{4}\left[f_i + 2 f_{i+\frac{1}{2}} + f_{i+1} \right]. \end{aligned} \tag{11a}$$

When the function f is monotone decreasing over (x_i, x_{i+1}), equation (11) reduces to

$$\begin{aligned} A_i^* &= \Delta f_{i+1} + \frac{\Delta}{2}\left[f_{i+\frac{1}{2}} - f_{i+1} \right] + \frac{\Delta}{4}\left[f_{i+\frac{1}{2}} - f_{i+1} \right] + \frac{\Delta}{4}\left[f_i - f_{i+\frac{1}{2}} \right] \\ &= \frac{\Delta}{4}\left[4 f_{i+1} + 2 f_{i+\frac{1}{2}} - 2 f_{i+1} + f_{i+\frac{1}{2}} - f_{i+1} + f_i - f_{i+\frac{1}{2}} \right] \\ &= \frac{\Delta}{4}\left[f_{i+1} + 2 f_{i+\frac{1}{2}} + f_i \right] \\ &= \frac{\Delta}{4}\left[f_i + 2 f_{i+\frac{1}{2}} + f_{i+1} \right]. \end{aligned} \tag{11b}$$

Clearly, the results are the same irrespective of whether f is monotone increasing or decreasing; hence result.

2.2 Proposed composite numerical integration method

The proposed composite method provides a formula for estimating numerically the area under the curve of f and above the horizontal axis between the intervals $[a,b]$. It is the sum of the areas of all the n strips each of width $\Delta = \frac{b-a}{n}$ and k sub-divisions at the top as indicated in the diagram (Figs. 1a and 1b). Thus the proposed composite numerical integration method is given by;

$$A_n = \frac{\Delta}{2k}\left[f_0 + 2\sum_{i=1}^{n-1} f_i + 2\sum_{i=0}^{n-1}\sum_{t=1}^{k-1} f_{i+\frac{t}{k}} + f_n \right]. \tag{12}$$

Where the subscript in A_n means the area estimation is based on n segments in the interval $[a, b]$. The symbols in (12) are as previously defined.

The derivation follows trivially by summing the $A_i^* \,(i = 0,1,2,...,n-1)$ in equation (10).

When $k = 2$, equation (12) reduces to

$$A_n = \frac{\Delta}{4}\left[f_0 + 2\sum_{i=1}^{n-1} f_i + 2\sum_{i=0}^{n-1} f_{i+\frac{1}{2}} + f_n \right].$$

When $k = 3,$ we have

$$A_n = \frac{\Delta}{6}\left[f_0 + 2\sum_{i=1}^{n-1} f_i + 2\sum_{i=0}^{n-1} f_{i+\frac{1}{3}} + 2\sum_{i=0}^{n-1} f_{i+\frac{2}{3}} + f_n \right].$$

The cases for $k \geq 4$ can similarly be generated.

Example

As an illustration, we will use the exact method and the various numerical methods mentioned to estimate the following integral:

$$\int_0^2 e^{x^2} dx$$

Method 1: "Exact" Integration

$$\begin{aligned}
\int_0^2 e^{x^2} dx &= \int_0^2 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} dx \\
&= \sum_{n=0}^{\infty} \int_0^2 \frac{x^{2n}}{n!} dx \\
&= \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)n!} \\
&= 2 + \frac{2^3}{3} + \frac{2^5}{5(2!)} + \frac{2^7}{7(3!)} + ... + \frac{2^{23}}{23(11!)}. \\
&=16.45263
\end{aligned}$$

Now to estimate the given integral by the various methods mentioned above, we choose $n = 12$ divisions of the interval [0,2] from which we obtain $\Delta = \frac{b-a}{12} = \frac{2-0}{12} = 0.1667$ together with consequent $x_i = a +$

$i\Delta$ values and the corresponding ordinates, $f_i = e^{x_i^2}$ $(i = 0, 1, 2, \ldots, 12)$ shown in Table 1 in the appendix section.

Method 2: Trapezium rule

The estimate of the integral by the trapezium rule is given by

$$= \frac{0.1667}{2}\left[1 + 2(1.0282 + 1.1175 + \ldots + 16.0832 + 28.8212) + 54.5982\right]$$

$$= \frac{1}{12}\left[1 + 2(73.9196) + 54.5982\right]$$

$$= \frac{203.4374}{12} = 16.95311\,.$$

Method 3: Simpson's $\frac{1}{3}$ rule

Using Simpson's $\frac{1}{3}$ rule, the estimate of the integral is given by

$$= \frac{0.16667}{3}\left[1 + 4(1.0282 + 1.2840 + 2.0026 + 3.9005 + 9.4877 + 28.8212) + \right.$$
$$\left. 2(1.1175 + 1.5596 + 2.7183 + 5.9167 + 16.0832) + 54.5982\right]$$

$$= \frac{1}{18}\left[1 + 4(46.5243) + 2(27.3954) + 54.5982\right]$$

$$= \frac{296.4859}{18} = 16.47144\,.$$

Method 4: Simpson's $\frac{3}{8}$ rule

By the Simpson's $\frac{3}{8}$ rule, the estimate of the integral is

$$= \frac{3(0.16667)}{8}\left[1 + 3(1.0282 + 1.1175 + 1.5596 + 2.0026 + 3.9005 + 5.9167 + \right.$$
$$\left. 16.0832 + 28.8212) + 2(1.2840 + 2.7183 + 9.477) + 54.5982\right]$$

$$= \frac{1}{16}\left[1 + 3(60.4296) + 2(13.4900) + 54.5982\right]$$

$$= \frac{263.8669}{16} = 16.49168\,.$$

Method 5: Boole's rule

Using Boole's rule, the estimate is

$$= \frac{2(0.16667)}{45}\left[7(1+54.5982)+32(1.0282+1.2840+...+9.4877+28.8212)+ 12(1.1175+2.7183+16.0832)+14(1.5596+5.9167)\right]$$

$$= \frac{1}{135}\left[7(55.5982)+32(46.5243)+12(19.91904)+14(7.4763)\right]$$

$$= \frac{2221.66}{135} = 16.45674\,.$$

Method 6: Weddle's rule

Using Weddle's rule, the estimate is

$$= \frac{3(0.16667)}{10}\left[1+5(1.0282+3.9005)+(1.1175+5.9167)+6(1.2840+9.4877)+ (1.5596+16.0832)+5(2.0026+28.8212)+2(2.7183)+54.5982\right]$$

$$= \frac{1}{20}\left[1+5(33.0187)+6(10.7717)+2(2.7183)+24.677+54.5982\right]$$

$$= \frac{329.1048}{20} = 16.45524\,.$$

Method 7: Proposed Numerical Integration Method

For this method, the values of x_i and $x_{i+\frac{1}{2}}$, together their corresponding ordinates $f_i = e^{x_i^2}$ and $f_{i+\frac{1}{2}} = e^{x_{i+\frac{1}{2}}}$, $i = 0, 1, \ldots 12$ are shown in Table 1 in the appendix. The values are based on $k = 2$ for the proposed method. As indicated early on, $k = 1$ is the same as the Trapezium rule. Thus by the proposed method for $k = 2$, an estimate of the integral is given as follows

$$= \frac{(0.1667)}{4}\left[1+2(1.0070+1.0282+1.0645+...+39.3939)+54.5982\right]$$

$$= \frac{1}{24}\left[1+2(171.1452)+54.5982\right]$$

$$= \frac{397.8886}{24} = 16.57869\,.$$

Generally, when $k = 8, 9, \ldots, 14$, the values for the Proposed Integration method for evaluating $\int_0^2 e^{x^2}dx$ taking 12 segments are given in Table 1.

Table 1. Estimates of the area using numerical integration method with 12 segments

Integration method	Area	True error	Absolute relative true errors, $\lvert\varepsilon_a\rvert$	m
Exact (Numerical integration)	16.452630	-		
Trapezium rule	16.953110	0.500480	3.0419%	1
Simpson's 1/3	16.471440	0.018810	0.1143%	3
Simpson's 3/8	16.491680	0.039050	0.2373%	2
Boole's rule	16.456740	0.004110	0.0250%	3
Weddle's rule	16.455240	0.002610	0.0159%	4
Proposed method when k=8	16.460530	0.007900	0.0480%	3
Proposed method when k=9	16.458870	0.006240	0.0379%	3
Proposed method when k=10	16.457680	0.005050	0.0307%	3
Proposed method when k=11	16.456810	0.004180	0.0254%	3
Proposed method when k=12	16.456140	0.003510	0.0213%	3
Proposed method when k=13	16.455620	0.002990	0.0182%	4
Proposed method when k=14	16.455210	0.002580	0.0157%	4
Proposed method when k=15	16.454870	0.002240	0.0136%	4
Proposed method when k=16	16.454600	0.001970	0.0120%	4
Proposed method when k=17	16.454380	0.001750	0.0106%	4
Proposed method when k=18	16.454190	0.001560	0.0095%	4
Proposed method when k=19	16.454030	0.001400	0.0085%	4
Proposed method when k=20	16.453890	0.001260	0.0077%	4

** m = the number of significant digits at least correct*
Source: own research

The entries in the last columns of Table 1 are obtained as follows.

$$\varepsilon_a = \frac{Exact - Approximate}{Exact}$$

$$m = 2 - \log\left(\frac{\lvert\varepsilon_a\rvert}{0.02}\right)$$

3 Results and Discussion

Comparing the Proposed Numerical Integration Method to the various numerical integration formulas from Table 1 above using the absolute relative true errors, when $k = 1$, the area under the curve is 16.95311 which is the same as the area under the curve using the Trapezium rule. Also when $k = 2$ and above, the estimates of the area under the curve are better than the Trapezium rule. That is, the Trapezium rule yields the highest absolute relative true error of 3.0419% as compared to the "exact" integration. Furthermore, when k = 6 and above, the estimates of the area under the curve are better than the Simpson $1/3$ and $3/8$ rule. When $k = 11$ and above, the estimates of the area under the curve are better than Boole's rule and when k = 13 and above, the estimates of the area under the curve are better than Weddle's rule. The proposed method with smaller divisions of the interval gives a better estimate with lesser errors as compared to the Trapezium rule, Simpson $1/3$ and $3/8$ rule, Boole's and Weddle's rule. Similar results for different number of segments are evident in Tables 2 to 7.

Given an absolute relative true error bound of $\leq 0.02\%$ clearly from Tables 1 to 7 as the Δ decrease, k decreases. Hence it suffices to determine the relationship between k and Δ so that for the function in (12), one can determine k the number of divisions at the top to arrive at the desired estimate with the stated absolute relative true error bound.

To find the equation, a scatter plot was drawn to find the relationship as shown in Fig. 2. It resulted in a regression equation of k on Δ as $k = 80.7240 \times \Delta$ with a coefficient of determination of 0.9892.

Hence, to estimate the area under the curve of our example with 20 segments, the number of divisions, k at the top of each segment required to obtain an absolute relative true error less than 0.02% is given by $k = 80.7240\left(\frac{2-0}{20}\right) \approx 8$.

Further research can be made on several groups of functions to show whether the same relationship exists as the function in the example so that when k is decreasing, Δ is decreasing.

4 Draw Backs

The Proposed Numerical Integration Method is simple and gives accurate estimates in definite integration but requires computing more ordinates than in the case of the other methods with the same number of segments. Nevertheless, with the help of computers, one can go around this drawback by developing algorithms which will do the computations with ease.

5 Conclusion

The proposed method with smaller segmentation give better estimates with smaller errors than the trapezium, Boole's, Weddle's and Simpson's rules. It also provided a formula for the number of divisions required at the top of each segment to obtain an estimate with an absolute relative true error less than a stated tolerance.

Competing Interests

Authors have declared that no competing interests exist.

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Appendix A

Table 2. Estimates of the area using numerical integration method with 8 segments

Integration method	Area	True error	Absolute relative true errors, $\lvert\varepsilon_a\rvert$	m
"Exact" Integration	16.45263	-		
Trapezium rule	17.56509	1.11246	6.76159%	1
Simpson's 1/3 rule	16.53859	0.08596	0.52247%	2
Simpson's 3/8 rule	-	-	-	-
Boole's rule	16.48426	0.03163	0.19225%	3
Weddle's rule	-	-	-	-
Proposed method when k=8	16.47039	0.01776	0.10795%	3
Proposed method when k=9	16.46667	0.01404	0.08534%	3
Proposed method when k=10	16.46400	0.01137	0.06911%	3
Proposed method when k=11	16.46203	0.00940	0.05713%	3
Proposed method when k=12	16.46053	0.00790	0.04802%	3
Proposed method when k=13	16.45936	0.00673	0.04091%	3
Proposed method when k=14	16.45843	0.00580	0.03525%	3
Proposed method when k=15	16.45768	0.00505	0.03069%	3
Proposed method when k=16	16.45707	0.00444	0.02699%	3
Proposed method when k=17	16.45656	0.00393	0.02389%	3
Proposed method when k=18	16.45614	0.00351	0.02133%	3
Proposed method when k=19	16.45578	0.00315	0.01915%	4
Proposed method when k=20	16.45547	0.00284	0.01726%	4

** m = the number of significant digits at least correct*
Source: own research

Table 3. Estimates of the area using numerical integration method with 9 segments

Integration method	Area	True error	Absolute relative true errors, $\lvert\varepsilon_a\rvert$	m
"Exact" Integration	16.45263	-		
Trapezium rule	17.33562	0.88299	5.36686%	1
Simpson's 1/3 rule	-	-	-	-
Simpson's 3/8 rule	16.56296	0.03163	0.19225%	3
Boole's rule	-	-	-	-
Weddle's rule	-	-	-	-
Proposed method when k=8	16.46667	0.01777	0.10801%	3
Proposed method when k=9	16.46372	0.01404	0.08534%	3
Proposed method when k=10	16.46161	0.01137	0.06911%	3
Proposed method when k=11	16.46005	0.00940	0.05713%	3
Proposed method when k=12	16.45887	0.00624	0.03793%	3
Proposed method when k=13	16.45795	0.00532	0.03234%	3
Proposed method when k=14	16.45721	0.00458	0.02784%	3
Proposed method when k=15	16.45662	0.00399	0.02425%	3
Proposed method when k=16	16.45614	0.00351	0.02133%	3
Proposed method when k=17	16.45574	0.00311	0.01890%	4
Proposed method when k=18	16.45540	0.00277	0.01684%	4
Proposed method when k=19	16.45512	0.00249	0.01513%	4
Proposed method when k=20	16.45487	0.00224	0.01361%	4

** m = the number of significant digits at least correct*
Source: own research

Table 4. Estimates of the area using numerical integration method with 10 segments

Integration method	Area	True error	Absolute relative true errors, $\lvert\varepsilon_a\rvert$	m
"Exact" Integration	16.45263	-		
Trapezium rule	17.17021	0.71758	4.36149%	1
Simpson's 1/3 rule	16.49020	0.03758	0.22841%	2
Simpson's 3/8 rule	-	-	-	-
Boole's rule	-	-	-	-
Weddle's rule	-	-	-	-
Proposed method when k=8	16.46400	0.01109	0.06741%	3
Proposed method when k=9	16.46161	0.00899	0.05464%	3
Proposed method when k=10	16.45991	0.00728	0.04425%	3
Proposed method when k=11	16.45864	0.00602	0.03659%	3
Proposed method when k=12	16.45768	0.00505	0.03069%	3
Proposed method when k=13	16.45693	0.00430	0.02614%	3
Proposed method when k=14	16.45634	0.00371	0.02255%	3
Proposed method when k=15	16.45586	0.00323	0.01963%	4
Proposed method when k=16	16.45547	0.00284	0.01726%	4
Proposed method when k=17	16.45515	0.00252	0.01532%	4
Proposed method when k=18	16.45487	0.00224	0.01361%	4
Proposed method when k=19	16.45464	0.00201	0.01222%	4
Proposed method when k=20	16.45445	0.00182	0.01106%	4

** m = the number of significant digits at least correct*
Source: own research

Table 5. Estimates of the area using numerical integration method with 11 segments

Integration method	Area	True error	Absolute relative true errors, $\lvert\varepsilon_a\rvert$	m
"Exact" integration	16.45263	-		
Trapezium rule	17.04713	0.59450	3.61340%	1
Simpson's 1/3 rule	-	-	-	-
Simpson's 3/8 rule	-	-	-	-
Boole's rule	-	-	-	-
Weddle's rule	-	-	-	-
Proposed method when k=8	16.46203	0.00940	0.05713%	3
Proposed method when k=9	16.46005	0.00743	0.04516%	3
Proposed method when k=10	16.45864	0.00602	0.03659%	3
Proposed method when k=11	16.45760	0.00497	0.03021%	3
Proposed method when k=12	16.45681	0.00418	0.02541%	3
Proposed method when k=13	16.45619	0.00356	0.02164%	3
Proposed method when k=14	16.45570	0.00307	0.01866%	4
Proposed method when k=15	16.45530	0.00267	0.01623%	4
Proposed method when k=16	16.45498	0.00235	0.01428%	4
Proposed method when k=17	16.45471	0.00208	0.01264%	4
Proposed method when k=18	16.45448	0.00185	0.01124%	4
Proposed method when k=19	16.45429	0.00166	0.01009%	4
Proposed method when k=20	16.45413	0.00150	0.00912%	4

** m = the number of significant digits at least correct*
Source: own research

Table 6. Estimates of the area using numerical integration method with 13 segments

Integration method	Area	True error	Absolute relative true errors, $\|\varepsilon_a\|$	m
"Exact" Integration	16.45263	-		
Trapezium rule	16.87970	0.42708	2.59582%	1
Simpson's 1/3 rule	-	-	-	-
Simpson's 3/8 rule	-	-	-	-
Boole's rule	-	-	-	-
Weddle's rule	-	-	-	-
Proposed method when k=8	16.45936	0.00673	0.04091%	3
Proposed method when k=9	16.45795	0.00532	0.03234%	3
Proposed method when k=10	16.45693	0.00431	0.02620%	3
Proposed method when k=11	16.45619	0.00356	0.02164%	3
Proposed method when k=12	16.45562	0.00299	0.01817%	4
Proposed method when k=13	16.45518	0.00255	0.01550%	4
Proposed method when k=14	16.45483	0.00220	0.01337%	4
Proposed method when k=15	16.45454	0.00191	0.01161%	4
Proposed method when k=16	16.45431	0.00168	0.01021%	4
Proposed method when k=17	16.45412	0.00149	0.00906%	4
Proposed method when k=18	16.45396	0.00133	0.00808%	4
Proposed method when k=19	16.45382	0.00119	0.00723%	4
Proposed method when k=20	16.45370	0.00107	0.00650%	4

** m = the number of significant digits at least correct*
Source: own research

Table 7. Estimates of the area using numerical integration method with 14 segments

Integration method	Area	True error	Absolute relative true errors, $\|\varepsilon_a\|$	m
"Exact" Integration	16.45263	-		
Trapezium rule	16.82130	0.36868	2.24086%	1
Simpson's 1/3 rule	16.46302	0.01039	0.06315%	3
Simpson's 3/8 rule	-	-	-	-
Boole's rule	-	-	-	-
Weddle's rule	-	-	-	-
Proposed method when k=8	16.45843	0.00580	0.03525%	3
Proposed method when k=9	16.45721	0.00458	0.02784%	3
Proposed method when k=10	16.45634	0.00371	0.02255%	3
Proposed method when k=11	16.45500	0.00307	0.01866%	4
Proposed method when k=12	16.45521	0.00258	0.01568%	4
Proposed method when k=13	16.45483	0.00220	0.01337%	4
Proposed method when k=14	16.45452	0.00189	0.01149%	4
Proposed method when k=15	16.45428	0.00165	0.01003%	4
Proposed method when k=16	16.45408	0.00145	0.00881%	4
Proposed method when k=17	16.45391	0.00128	0.00778%	4
Proposed method when k=18	16.45377	0.00114	0.00693%	4
Proposed method when k=19	16.45366	0.00103	0.00626%	4
Proposed method when k=20	16.45356	0.00093	0.00565%	4

** m = the number of significant digits at least correct*
Source: own research

Table A1. Abscissas x_i , $x_{i+\frac{1}{2}}$ and corresponding ordinates f_i , $f_{i+\frac{1}{2}}$

i	x_i	f_i	$x_{i+\frac{1}{2}}$	$f_{i+\frac{1}{2}}$
0	0	1	0.0833	1.0070
1	0.1667	1.0282	0.2500	1.0645
2	0.3333	1.1175	0.4167	1.1896
3	0.5000	1.2840	0.5833	1.4053
4	0.6667	1.5596	0.7500	1.7551
5	0.8333	2.0026	0.9167	2.3170
6	1.0000	2.7183	1.0833	3.2336
7	1.1667	3.9005	1.2500	4.7707
8	1.3333	5.9167	1.4167	7.4405
9	1.5000	9.4877	1.5833	12.2674
10	1.6667	16.0832	1.2750	21.3809
11	1.8333	28.8212	1.9167	39.3939
12	2.0000	54.5982		

Source: own research

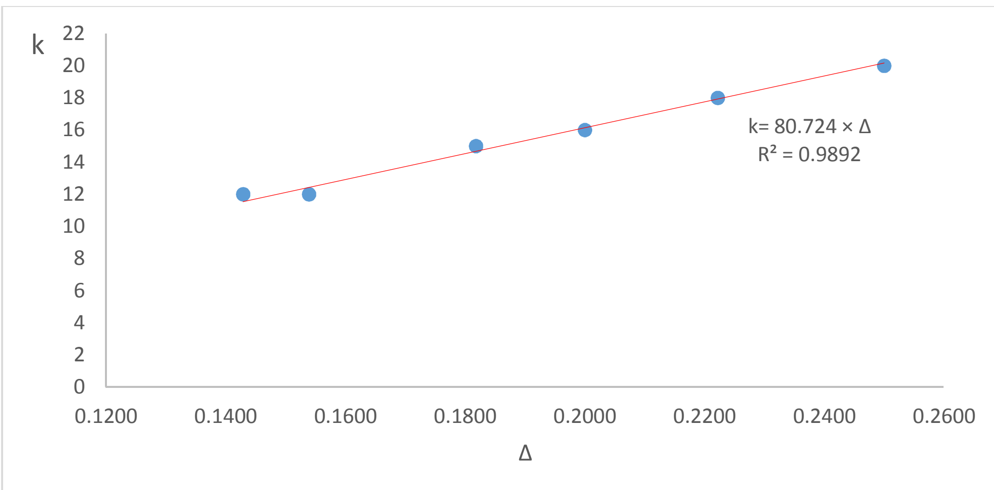

Fig. 2. A regression plot of k on Δ
Source: own research

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