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On conics and their tangents

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Abstract: We present, in a way quite accessible to undergraduate and graduate students, some basic and important facts about conics: parabola, ellipse and hyperbola. For each conic, we start by its definition, then consider tangent line and obtain an elementary proof of the reflexion property. We study intersection of tangents. We obtain the orthopic set for orthogonal tangents: the directrix for parabola and the Monge's circle for ellipse and hyperbola. For ellipse and hyperbola we also consider intersection of tangents for parallel rays at points of intersection with the conic. Those analysis lead to geometric methods to draw conics. Finally we get the directrices for ellipse and hyperbola by considering intersections of tangents at endpoints of a secant passing through a focus.

Keywords: Conics, reflexion property, tangent, orthoptic set, principal circle, directrix.

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1. Introduction

Conics are defined as locus which is, in general in geometry, a set of all points (commonly, a line, a line segment, a curve or a surface), whose location satisfies or is determined by one or more specified conditions. There are many real world applications of conics, some of them were already presented in [1–3], and more recently in [4–6]. The study of conics is a part of the mathematical field called *Analytic Geometry*, see [7,8] for more details, and for example Monge's work is a quite important contribution to this subject [9]. It is an old elementary and interesting mathematical subject. Interested readers who would like to learn more about this subject are referred to the preceding references and to the following books [10–14], and references therein.

In this paper we prove facts related to tangents to conics using very elementary techniques. So this text is quite accessible to undergraduate and graduate students. Our paper is based on cartesian coordinate systems and proceed with the coordinates of points and vectors. For each type of conics, namely parabola, ellipse and hyperbola, we start with its definition an get its cartesian equation. Then we consider tangents to conic. We follow by given an elementary and direct proof of its reflexion property. We continue by considering intersection of tangents. In particular, for the intersection of orthogonal tangents, we obtain the orthoptic set of the conic. The orthoptic (curve) is the locus of the points by which pass two perpendicular tangents to the curve, in other words, the locus of the points from which we"see" the curve under a right angle. For a parabola it is its directrix, and for ellipse and hyperbola it is known as the Monge's circle. For ellipse and hyperbola we continue by considering intersection of tangents at the two points of intersenction of the conic with two parallel rays passing through the foci, so we get the principal circle of the conic. Finally we consider intersection of tangents at the two points of intersection of a ray passing through one focus, we get the directrices of the conics.

2. Notations and preliminaries

For two points $P = (x_p, y_p)$ and $Q = (x_q, y_q)$ in $\mathbb{R}^2$ seen as vectors, we use the scalar product

$$P \cdot Q = x_p x_q + y_p y_q,$$

and the length

$$|P| = \sqrt{P \cdot P} = \sqrt{x_p^2 + y_p^2}.$$

Also, we use the following useful relation

$$P \cdot Q = |P|\,|Q|\cos(\theta),$$

where θ is the angle between the two vectors P and Q.

Seen as point in $\mathbb{R}^2$, the unique line determined by the two points P and Q is noted $\overleftrightarrow{PQ}$. The set of points on this line between P and Q is the segment

$$\overline{PQ} = \left\{ R = (1-\lambda)P + \lambda Q \in \mathbb{R}^2 \ : \ \lambda \in [0,1] \right\}.$$

Also PQ is the vector $PQ = Q - P$. The length of the vector PQ, or equivalently the length of the segment $\overline{PQ}$, is noted $|PQ| = |Q - P|$.

A line ℓ through the point P in a direction $\nu = (\nu_x, \nu_y)$ is defined by

$$\ell = P + [\nu] = \left\{ P_\lambda = P + \lambda\nu \in \mathbb{R}^2 \ : \ \lambda \in \mathbb{R} \right\}.$$

We use the notation $\nu^\perp = (-\nu_y, \nu_x)$ for a vector which is perpendicular or orthogonal to ν. It is also called the normal vector to the line ℓ. The line ℓ can also be written as

$$\ell = \left\{ Q \in \mathbb{R}^2 \ : \ (Q - P) \cdot \nu = 0 \right\}.$$

The orthogonal or perpendicular line to ℓ at P is then

$$\ell^\perp = P + \left[\nu^\perp\right] = \left\{ P_\lambda = P + \lambda\nu^\perp \in \mathbb{R}^2 \ : \ \lambda \in \mathbb{R} \right\}.$$

Finally the distance $|P\ell|$ from a point P to a line ℓ is defined by

$$|P\ell| = \min\left\{ |PQ| \ : \ Q \in \ell \right\}.$$

In this text we focus on tangents to conics. To determine the cartesian equation of a tangent we need its direction. This direction is obtained by computing the limit of direction of secants to the conic.

3. Parabola

3.1. Definition

A parabola is defined as the locus of a point which moves so that it is always the same distance from a fixed point F (called the focus) and a given line ℓ_d (called the directrix).

Let $p > 0$, the focus be $F = (p/2, 0)$, and the directrix ℓ_d be

$$\ell_d = -F + [(0,1)] = \left\{ (-p/2, y) \in \mathbb{R}^2 \ : \ y \in \mathbb{R} \right\}.$$

For a point $P = (x, y)$, we have

$$|FP| = |(x - p/2, y)| = \sqrt{(x - p/2)^2 + y^2},$$

and

$$\begin{aligned} |P\ell_d| &= \min\left\{ |PQ| \ : \ Q \in \ell_d \right\} \\ &= \min\left\{ |(x + p/2, y - \lambda)| \ : \ \lambda \in \mathbb{R} \right\} \\ &= |x + p/2|. \end{aligned}$$

The point P is on the parabola if and only if $|FP| = |P\ell_d|$. This condition leads to the equation

$$y^2 = 2px.$$

So the parabola is the set

$$\mathcal{P} = \left\{ P \in \mathbb{R}^2 \ : \ |FP| = |P\ell_d| \right\} = \left\{ P = (x,y) \in \mathbb{R}^2 \ : \ y^2 = 2px \right\}.$$

The x-axis, which is the horizontal line of equation $y = 0$, is the axis of symmetry of the parabola $\mathcal{P}$.

3.2. Tangent

Let us consider $P_h = (x_h, y_h) = \big((y+h)^2/2p, y+h\big) \in \mathcal{P}$ for $h \in \mathbb{R}$. For $h \neq 0$, the direction of the secant $\overleftrightarrow{P_0 P_h}$ is

$$\tau_h = \frac{1}{h}\left(P_h - P_0\right) = \left(\frac{y}{p} + \frac{h}{2p}, 1\right).$$

Then let $h \to 0$, so we get $\tau_0 = (y/p, 1)$. Then let us set $\tau = p\tau_0 = (y, p)$ to be the direction of the tangent to $\mathcal{P}$ at $P_0 = P = (x, y)$. So this tangent line is

$$\ell_t = P + [\tau] = \left\{ Q \in \mathbb{R}^2 \ : \ (Q - P) \cdot \nu = 0 \right\}.$$

where $\nu = \tau^{\perp} = (-p, y_0)$, which leads to the equation

$$-px + y_0 y = px_0.$$

3.3. Reflexion property

Let $P = (x, y) \in \mathcal{P}$ and $Q = (-p/2, y) \in \ell_d$, so we have $|FP| = |QP|$. Let α_1 be the angle between FP and the tangent ℓ_t, and let α_2 be the angle between QP and the tangent ℓ_t. If $\alpha_1 = \alpha_2$, an horizontal ray is reflected on the parabola through the focus F, this is the reflexion property of the parabola.

To show this property, let us observe that we have

$$\begin{cases} FP \cdot \tau &= |FP|\,|\tau| \cos(\alpha_1), \\ QP \cdot \tau &= |QP|\,|\tau| \cos(\alpha_2). \end{cases}$$

Moreover

$$\begin{cases} FP &= P - F &= (x - p/2, y), \\ QP &= P - Q &= (x + p/2, 0), \end{cases}$$

so

$$\begin{cases} FP \cdot \tau &= (x - p/2, y) \cdot (y, p) &= (x + p/2)y, \\ PQ \cdot \tau &= (x + p/2, 0) \cdot (y, p) &= (x + p/2)y, \end{cases}$$

from which we can conclude that $\cos(\alpha_1) = \cos(\alpha_2)$, or $\alpha_1 = \alpha_2$.

3.4. Intersection of tangents

3.4.1. General case

Let $P_i = (x_i, y_i) \in \mathcal{P}$ for $i = 1, 2$, be two points on the parabola. The common points to the two tangents to the parabola at those points P_i is the solution $\tilde{P} = (\tilde{x}, \tilde{y})$ to the linear system

$$\begin{bmatrix} -p & y_1 \\ -p & y_2 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = p \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which is

$$\tilde{P} = \left(\frac{y_1 y_2}{2p}, \frac{y_1 + y_2}{2}\right).$$

Let us observe that this point is on the horizontal line $y = \frac{y_1+y_2}{2}$ as $\bar{P} = \left(P_1 + P_2\right)/2 = \left(\left(x_1 + x_2\right)/2, \left(y_1 + y_2\right)/2\right)$.

3.4.2. Equally y-spaced P_i

Suppose $P_2 = (x_2, y_2) = \left((y_1+h)^2/2p, y_1+h\right) \in \mathcal{P}$. Then

$$\begin{cases} y_2 - y_1 &= h, \\ y_2 + y_1 &= 2\tilde{y}, \end{cases}$$

from which we obtain $y_1 y_2 = \frac{4\tilde{y}^2 - h^2}{4}$. Since $\tilde{x} = \frac{y_1 y_2}{2p}$, so

$$\tilde{y}^2 = 2p\left(\tilde{x} + \frac{h^2}{8p}\right),$$

which is a second parabola, in fact it is the translated original parabola. Its directrix is $x = -\frac{p}{2} - \frac{h^2}{8p}$ and its focus is $\tilde{F} = \left(\frac{p}{2} - \frac{h^2}{8p}, 0\right)$.

3.4.3. Orthogonal tangents and orthoptic set

The orthogonality of the two tangents at $P_i = (x_i, y_i) \in \mathcal{P}$ for $i = 1, 2$, means that

$$\tau_1 \cdot \tau_2 = 0 \text{ or } \nu_1 \cdot \nu_2 = 0 \text{ or } \text{ or } y_1 y_2 + p^2 = 0.$$

In this case $\tilde{x} = -p/2$ and $\tilde{P}$ is on the directrix ℓ_d. Moreover any point on the directrix $x = -p/2$ is a point in the orthoptic set. Indeed, remember that $\tilde{y} = \frac{y_1+y_2}{2}$, so if we set $\tilde{y} = \zeta$, and solve the system

$$\begin{cases} y_1 + y_2 &= 2\zeta, \\ y_1 y_2 &= -2p^2, \end{cases}$$

we get two values $\zeta \pm \sqrt{\zeta^2 + p^2}$ for y_1 and y_2, in fact the y coordinates of the corresponding two points P_1 and P_2 on the parabola. In conclusion, the orthoptic curve to the parabola is its directrix.

The orthogonality condition of tangent also means that the focus F is on the secant $\overleftrightarrow{P_1P_2}$. To see this fact, consider $FP_1 = (x_1 - p/2, y_1)$ and $FP_2 = (x_2 - p/2, y_2)$. Then $F \in \overleftrightarrow{P_1P_2}$ if and only if the angle between FP_1 and FP_2 is 0 or π, or the angle between FP_1 and $FP_2^\perp$ is $\pi/2$ or $3\pi/2$, which means that $FP_1 \cdot FP_2^\perp = 0$. But

$$FP_1 \cdot FP_2^\perp = (x_1 - p/2, y_1) \cdot (-y_2, x_2 - p/2) = \frac{y_1 y_2 + p^2}{2p}(y_2 - y_1).$$

So, since $y_1 \neq y_2$ because the P_i's are not on an horizontal line, F being on $\overleftrightarrow{P_1P_2}$ leads to the orthogonality condition $y_1 y_2 + p^2 = 0$, and conversely.

3.5. A geometric construction

Let $Q = (-p/2, y) \in \ell_d$. Let R be the middle point of $\overline{QF}$,

$$R = \frac{1}{2}(Q + F) = (0, y/2).$$

Consider now the line ℓ_r passing through R and perpendicular to $\overleftrightarrow{QF}$. Since

$$QF = F - Q = (p, -y) \quad \text{and} \quad QF^\perp = (y, p)$$

then

$$\ell_r = R + \left[QF^\perp\right] = R + [(y, p)].$$

The point of intersection of ℓ_r and the horizontal line $\ell_h = Q + [(1, 0)]$ is the point $P = \left(y^2/2p, y\right) \in \mathcal{P}$. So this is in fact a geometric method to construct the parabola.

4. Ellipse

4.1. Definition

An ellipse is a plane curve surrounding two points called foci, such that for all points on the curve, the sum of the two distances to the focal points is a constant.

Let $a > c > 0$ and define $b = \sqrt{a^2 - c^2}$. Let $F_1 = (c, 0)$ and $F_2 = (-c, 0)$ be the foci of the ellipse. For a point $P = (x, y)$, we have $F_1P = P - F_1 = (x - c, y)$ and $F_2P = P - F_2 = (x + c, y)$. A point P is on the ellipse if and only if

$$|F_1P| + |F_2P| = 2a.$$

This condition leads to the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and the ellipse is the set

$$\mathcal{E} = \left\{P \in \mathbb{R}^2 \ : \ |F_1P| + |F_2P| = 2a\right\} = \left\{P = (x, y) \in \mathbb{R}^2 \ : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}.$$

The horizontal x-axis and the vertical y-axis are axes of symmetry of the ellipse. As a consequence, the ellipse is also symmetric with respect to the origin $O = (0, 0)$.

Finally using the parametrization $P = (x, y) = (a\cos(\theta), b\sin(\theta))$ for $\theta \in \mathbb{R}$, we have $P \in \mathcal{E}$. The coordinates of P can be obtained by considering two circles of radius respectively a and b, which leads to a geometric construction of an ellipse.

4.2. Tangent

Without loosing in generality, let us consider the figure in the right half space $x > 0$. Let $P_h = (x_h, y_0 + h) \in \mathcal{E}$ on the ellipse for h such that $y_0 + h \in (-b, b)$. Let $h \neq 0$, the direction of the secant $\overleftrightarrow{P_0P_h}$ to the ellipse is

$$\begin{aligned}
\tau_h &= \frac{1}{h}\left(P_h - P_0\right) \\
&= \left(\frac{a}{h}\left[\sqrt{1 - \frac{(y_0 + h)^2}{b^2}} - \sqrt{1 - \frac{y_0^2}{b^2}}\right], 1\right) \\
&= \left(-\frac{a}{b^2}\frac{2y_0 + h}{\sqrt{1 - \frac{(y_0+h)^2}{b^2}} + \sqrt{1 - \frac{y_0^2}{b^2}}}, 1\right).
\end{aligned}$$

Then let $h \to 0$, so we get $\tau_0 = \left(-\frac{y_0/b^2}{x_0/a^2}, 1\right)$. Then we set $\tau = -\frac{x_0}{a^2}\tau_0 = \left(y_0/b^2, -x_0/a^2\right)$ to be the direction of the tangent line to $\mathcal{E}$ at $P = P_0$. So this tangent line is

$$\ell_t = P + [\tau] = \left\{Q \in \mathbb{R}^2 \ : \ (Q - P) \cdot \nu = 0\right\},$$

for $\nu = \tau^\perp = (x_0/a^2, y_0/b^2)$, which leads to the equation

$$\frac{x_0}{a^2}x + \frac{y_0}{b^2}y = 1.$$

4.3. Reflexion property

Let $P = (x, y) \in \mathcal{E}$, we have

$$\begin{cases} PF_1 \cdot \tau &= |PF_1|\,|\tau|\cos(\alpha_1), \\ F_2P \cdot \tau &= |F_2P|\,|\tau|\cos(\alpha_2). \end{cases}$$

where α_i $(i = 1, 2)$ are respectively the angles between PF_1, and F_2P, and the tangent vector τ. To prove the reflexion property we have to prove that $\alpha_1 = \alpha_2$.

We have

$$\begin{cases} PF_1 & = & F_1 - P & = & (c-x,-y), \\ F_2P & = & P - F_2 & = & (x+c,y). \end{cases}$$

So

$$\begin{cases} |PF_1| & = & a\left(1-\frac{c}{a^2}x\right), \\ |F_2P| & = & a\left(1+\frac{c}{a^2}x\right), \end{cases}$$

and

$$\begin{cases} PF_1\cdot\tau & = & (c-x,-y)\cdot(y/b^2,-x/a^2) & = & \frac{cy}{b^2}\left(1-\frac{c}{a^2}x\right), \\ F_2P\cdot\tau & = & (x+c,y)\cdot(y/b^2,-x/a^2) & = & \frac{cy}{b^2}\left(1+\frac{c}{a^2}x\right). \end{cases}$$

So we can conclude that $\cos(\alpha_1)=\cos(\alpha_2)$, or $\alpha_1=\alpha_2$, which is a proof of the reflexion property of the ellipse which means that a ray passing through one focus is reflected on the ellipse into a ray passing through the other focus.

4.4. Intersection of tangents

4.4.1. General case

Let $P_i=(x_i,y_i)\in\mathcal{E}$ for $i=1,2$, be two points on the ellipse such that $\bar{P}=\frac{1}{2}(P_1+P_2)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)\neq O$, or equivalently that $O\notin\overleftrightarrow{P_1P_2}$. This condition means that

$$P_2\cdot P_1^{\perp}=(x_2,y_2)\cdot(-y_1,x_1)=x_1y_2-x_2y_1\neq 0.$$

The common points to the two tangents to the ellipse at those points P_i is the solution $\tilde{P}=(\tilde{x},\tilde{y})$ to the linear system

$$\begin{bmatrix} \frac{x_1}{a^2} & \frac{y_1}{b^2} \\ \frac{x_2}{a^2} & \frac{y_2}{b^2} \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which is

$$\tilde{P}=\frac{1}{x_1y_2-x_2y_1}\left(a^2(y_2-y_1),b^2(x_1-x_2)\right).$$

Let us observe that this point is also on the line passing through O and $\bar{P}$. Indeed

$$\tilde{P}\cdot\bar{P}^{\perp}=\frac{1}{x_1y_2-x_2y_1}\left(a^2(y_2-y_1),b^2(x_1-x_2)\right)\cdot\left(-\frac{y_1+y_2}{2},\frac{x_1+x_2}{2}\right)=0.$$

In fact

$$\tilde{P}=\frac{1}{x_1y_2-x_2y_1}\left(a^2(y_2-y_1),b^2(x_1-x_2)\right)=\tilde{\rho}\left((x_1+x_2)/2,(y_1+y_2)/2\right),$$

where

$$\tilde{\rho}=\frac{4}{\frac{(x_1+x_2)^2}{a^2}+\frac{(y_1+y_2)^2}{b^2}}.$$

4.4.2. Orthogonal tangents and orthoptic set

The orthogonality of the two tangents at $P_i\in\mathcal{E}$ $(i=1,2)$, means that

$$\tau_1\cdot\tau_2=0 \text{ or } \nu_1\cdot\nu_2=0 \text{ or } \frac{x_1x_2}{a^4}+\frac{y_1y_2}{b^4}=0.$$

To find the intersection $\tilde{P}=(\tilde{x},\tilde{y})$ of the two tangents, we have to solve the linear system

$$\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Using the orthogonality property of the two normal vectors ν_1 and ν_2, we get

$$\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\nu_1^t}{|\nu_1|^2} & \frac{\nu_2^t}{|\nu_2|^2} \end{bmatrix},$$

and then

$$\tilde{P} = (\tilde{x}, \tilde{y}) = \frac{1}{|\nu_1|^2}\nu_1 + \frac{1}{|\nu_2|^2}\nu_2.$$

Using the orthogonality condition again we get

$$\tilde{x}^2 + \tilde{y}^2 = |\tilde{P}|^2 = \tilde{P} \cdot \tilde{P} = \frac{1}{|\nu_1|^2} + \frac{1}{|\nu_2|^2}.$$

Now for the inverse matrix of the linear system, we have

$$I = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \begin{bmatrix} \frac{\nu_1^t}{|\nu_1|^2} & \frac{\nu_2^t}{|\nu_2|^2} \end{bmatrix},$$

and also

$$\begin{aligned} I &= \begin{bmatrix} \frac{\nu_1^t}{|\nu_1|^2} & \frac{\nu_2^t}{|\nu_2|^2} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \\ &= \frac{1}{|\nu_1|^2}\nu_1^t\nu_1 + \frac{1}{|\nu_2|^2}\nu_2^t\nu_2. \end{aligned}$$

Let us observe that

$$\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} I \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \sum_{i=1}^{2} \frac{1}{|\nu_i|^2} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \nu_i^t \nu_i \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

and taking the trace on both sides, we get

$$a^2 + b^2 = \sum_{i=1}^{2} \frac{1}{|\nu_i|^2} \left[a^2 \frac{x_i^2}{a^4} + b^2 \frac{y_i^2}{b^4} \right] = \frac{1}{|\nu_1|^2} + \frac{1}{|\nu_2|^2}.$$

So we have

$$\tilde{x}^2 + \tilde{y}^2 = a^2 + b^2,$$

and the point $\tilde{P}$ is on the circle of radius $\sqrt{a^2+b^2}$.

Moreover any point on this circle is the intersection of two orthogonal tangents to the ellipse. To see this fact, consider a point Q on this circle and $Q(\lambda) = \lambda Q$ for $\lambda \in (\lambda_0, +\infty)$, where λ_0 is such that $Q(\lambda_0) = \lambda_0 Q \in \mathcal{E}$. For any $Q(\lambda)$ we can find two tangents to the ellipse, and the angle between the tangents decreases continuously from π to 0 for λ increasing in $(\lambda_0, +\infty)$. So for a point $Q(\lambda)$ we have a right angle and the two tangents are perpendicular. From the preceding computation this point is on the circle of radius $\sqrt{a^2+b^2}$ and coincide with $Q = Q(\lambda) = \lambda Q$ so $\lambda = 1$.

In conclusion, the orthoptic set is the circle of radius $\sqrt{a^2+b^2}$, called the *Monge's circle*.

4.4.3. Parallel rays

We consider two parallel rays F_1P_1 and F_2P_2, with $P_i \in \mathcal{E}$ for $i = 1, 2$, and find the intersection of the tangents to the ellipse at those two points P_i.

We consider a point $F = (\sigma, 0)$ with $|\sigma| < a$, a direction $\eta = (a\cos(\theta), b\sin(\theta))$, and rewrite

$$P = F + \lambda\eta = (c, 0) + \lambda(a\cos(\theta), b\sin(\theta)).$$

The condition $P \in \mathcal{E}$ leads to the following values of λ

$$\lambda = \lambda_{\pm}(\sigma) = -\frac{\sigma}{a}\cos(\theta) \pm \sqrt{\left(1-\left(\frac{\sigma}{a}\right)^2\right) + \left(\frac{\sigma}{a}\right)^2 \cos^2(\theta)}.$$

For our purpose, let us use

$$\lambda_{+}(\sigma) = -\frac{\sigma}{a}\cos(\theta) + \sqrt{\left(1-\left(\frac{\sigma}{a}\right)^2\right) + \left(\frac{\sigma}{a}\right)^2 \cos^2(\theta)} > 0$$

and set

$$\begin{cases} P_1 &= F_1 + \lambda_{+}(c)\,(a\cos(\theta), b\sin(\theta)) &= (x_1, y_1), \\ P_2 &= F_2 + \lambda_{+}(-c)\,(a\cos(\theta), b\sin(\theta)) &= (x_2, y_2), \end{cases}$$

such that P_1 and P_2 are in the upper half space $(y > 0)$. The intersection of the tangents to the ellipse at those points is

$$\tilde{P} = (\tilde{x}, \tilde{y}) = \frac{1}{\sqrt{\left(1-\left(\frac{c}{a}\right)^2\right) + \left(\frac{c}{a}\cos(\theta)\right)^2}}\,(a\cos(\theta), b\sin(\theta))\,,$$

and we observe that

$$\tilde{x}^2 + \tilde{y}^2 = a^2.$$

So $\tilde{P}$ is a point on the *principal circle* of the ellipse, namely the circle centered at the origin O of radius a.

We also observe that $F_2\tilde{P}$ is perpendicular to the tangent to $\mathcal{E}$ at P_1. In fact a direct computation leads to

$$F_2\tilde{P} \cdot \tau_1 = (\tilde{P} - F_2) \cdot \left(y_1/b^2, -x_1/a^2\right) = 0.$$

In the same way $F_1\tilde{P}$ is perpendicular to the tangent to $\mathcal{E}$ at P_2, and also directly we verify that

$$F_1\tilde{P} \cdot \tau_2 = (\tilde{P} - F_1) \cdot \left(y_2/b^2, -x_2/a^2\right) = 0.$$

Those observations lead to a geometric construction of an ellipse by rotating a rectangle. Draw a rectangle such that two parallel sides pass through the foci with vertices on the circle of radius a. The intersections of lines passing through the foci and parallel to the diagonals of the rectangle intersect sides of the rectangle at points on $\mathcal{E}$. So by rotating the rectangle we can find $\mathcal{E}$ pointwise.

4.4.4. Tangents to endpoint of a secant

Using the notation of the preceding section, for any $|\sigma| < a$ and $\theta \in (0, \pi)$, we consider

$$\lambda_{\pm}(\sigma) = -\frac{\sigma}{a}\cos(\theta) \pm \sqrt{\left(1-\left(\frac{\sigma}{a}\right)^2\right) + \left(\frac{\sigma}{a}\right)^2 \cos^2(\theta)},$$

and, considering the focus F_1, we set

$$\begin{cases} P_1 &= F_1 + \lambda_{+}(c)(a\cos(\theta), b\sin(\theta)) &= (x_1, y_1), \\ P_2 &= F_1 + \lambda_{-}(c)(a\cos(\theta), b\sin(\theta)) &= (x_2, y_2). \end{cases}$$

In this way, the focus F_1 is on the secant $\overleftrightarrow{P_1P_2}$. The point of intersection $\tilde{P}$ of the tangents to the ellipse at P_i, $(i = 1, 2)$, is given by

$$\tilde{P} = (\tilde{x}, \tilde{y}) = \left(a^2/c, -b^2\cot(\theta)/c\right),$$

which means that $\tilde{P}$ is on the vertical line $x = a^2/c$. By symmetry, for the focus F_2 we get the line $x = -a^2/c$. Those two vertical lines $x = a^2/c$ and $x = -a^2/c$ are also called the *directrices* of the ellipse.

5. Hyperbola

5.1. Definition

A hyperbola is a set of points such that for any point of the set the absolute difference of the distances from two focal points, called foci, is a constant.

Let $c > a > 0$ and define $b = \sqrt{c^2 - a^2}$. Let $F_1 = (c, 0)$ and $F_2 = (-c, 0)$ be the foci of the hyperbola. For a point $P = (x, y)$, we have $F_1P = P - F_1 = (x - c, y)$ and $F_2P = P - F_2 = (x + c, y)$. A point P is on the hyperbola if and only if

$$\big|\, |F_1P| - |F_2P| \,\big| = 2a.$$

This condition leads to the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and the hyperbola is the set

$$\mathcal{H} = \Big\{ P \in \mathbb{R}^2 \;:\; \big|\, |F_1P| - |F_2P| \,\big| = 2a \Big\} = \Big\{ P = (x, y) \in \mathbb{R}^2 \;:\; \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Big\}.$$

The graph of the hyperbola in $\mathbb{R}^2$ has two branches: a first one in the right half space $x > 0$, and a second one in the left half space $x < 0$. The horizontal x-axis and the vertical y-axis are axes of symmetry of the hyperbola. Consequently the hyperbola is symmetric with respect to the origin $O = (0, 0)$.

The natural parametrization for the hyperbola is $P = (x, y) = (a\cosh(\theta), b\sinh(\theta))$ for $\theta \in \mathbb{R}$. This parametrization does not give us a geometric construction of the hyperbola.

5.2. Tangent

Without loosing in generality, let us consider the branch of the hyperbola in right half space ($x > 0$). Let us consider $P_h = (x_h, y_0 + h) \in \mathcal{H}$ for $h \in \mathbb{R}$. Let $h \neq 0$, the direction of the secant $\overleftrightarrow{P_0P_h}$ to the hyperbola is

$$\begin{aligned}
\tau_h &= \frac{1}{h}(P_h - P_0) \\
&= \left(\frac{a}{h}\left[\sqrt{1 + \frac{(y_0 + h)^2}{b^2}} - \sqrt{1 + \frac{y_0^2}{b^2}} \right], 1 \right) \\
&= \left(\frac{a}{b^2} \frac{2y_0 + h}{\sqrt{1 + \frac{(y_0+h)^2}{b^2}} + \sqrt{1 + \frac{y_0^2}{b^2}}}, 1 \right).
\end{aligned}$$

Then let $h \to 0$, so we get $\tau_0 = \left(\frac{y_0/b^2}{x_0/a^2}, 1 \right)$. Then we set $\tau = \frac{x_0}{a^2}\tau_0 = (y_0/b^2, x_0/a^2)$ to be the direction of the tangent line to $\mathcal{E}$ at $P = P_0$. Also, the tangent line is

$$\ell_t = P + [\tau] = \Big\{ Q \in \mathbb{R}^2 \;:\; (Q - P) \cdot \nu = 0 \Big\},$$

where $\nu = \tau^{\perp} = (-x_0/a^2, y_0/b^2)$, which leads to the equation

$$\frac{x_0}{a^2}x - \frac{y_0}{b^2}y = 1.$$

5.3. Reflexion property

Let $P = (x, y) \in \mathcal{H}$, we have

$$\begin{cases} F_1P \cdot \tau &= |F_1P|\,|\tau| \cos(\alpha_1), \\ F_2P \cdot \tau &= |F_2P|\,|\tau| \cos(\alpha_2). \end{cases}$$

where α_i is the angle between F_iP and the tangent vector τ for $i = 1, 2$. To prove the reflexion property we have to prove that $\alpha_1 = \alpha_2$.

We have

$$\begin{cases} F_1P &= P - F_1 &= (x-c, y), \\ F_2P &= P - F_2 &= (x+c, y). \end{cases}$$

So

$$\begin{cases} |F_1P| &= a\left(\frac{c}{a^2}x - 1\right), \\ |F_2P| &= a\left(\frac{c}{a^2}x + 1\right), \end{cases}$$

and

$$\begin{cases} F_1P \cdot \tau &= (x-c, y)\cdot(y/b^2, x/a^2) &= \frac{cy}{b^2}\left(\frac{c}{a^2}x - 1\right), \\ F_2P \cdot \tau &= (x+c, y)\cdot(y/b^2, x/a^2) &= \frac{cy}{b^2}\left(\frac{c}{a^2}x + 1\right). \end{cases}$$

So we can conclude that $\cos(\alpha_1) = \cos(\alpha_2)$, or $\alpha_1 = \alpha_2$, which is a proof of the reflexion property of the ellipse. A ray arriving in the direction of one focus is reflected on the hyperbola towards the other focus.

5.4. Intersection of tangents

5.4.1. General case

Let $P_i = (x_i, y_i) \in \mathcal{H}$, for $i = 1, 2$, be two points on the hyperbola such that $\bar{P} = \frac{1}{2}(P_1 + P_2) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \neq O$, or equivalently that $O \notin \overleftrightarrow{P_1P_2}$. This condition means that

$$P_2 \cdot P_1^{\perp} = (x_2, y_2) \cdot (-y_1, x_1) = x_1y_2 - x_2y_1 \neq 0.$$

The common points to the two tangents to the hyperbola at those points P_i is the solution $\tilde{P} = (\tilde{x}, \tilde{y})$ to the linear system

$$\begin{bmatrix} \frac{x_1}{a^2} & -\frac{y_1}{b^2} \\ \frac{x_2}{a^2} & -\frac{y_2}{b^2} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which is

$$\tilde{P} = \frac{1}{x_1y_2 - x_2y_1}\left(a^2(y_2 - y_1), b^2(x_2 - x_1)\right).$$

Let us observe that this point is also on the line passing through O and $\bar{P}$. Indeed

$$\tilde{P} \cdot \bar{P}^{\perp} = \frac{1}{x_1y_2 - x_2y_1}\left(a^2(y_2 - y_1), b^2(x_2 - x_1)\right) \cdot \left(-\frac{y_1+y_2}{2}, \frac{x_1+x_2}{2}\right) = 0.$$

In fact

$$\tilde{P} = \frac{1}{x_1y_2 - x_2y_1}\left(a^2(y_2 - y_1), b^2(x_2 - x_1)\right) = \tilde{\rho}\left((x_1+x_2)/2, (y_1+y_2)/2\right),$$

where

$$\tilde{\rho} = \frac{4}{\frac{(x_1+x_2)^2}{a^2} - \frac{(y_1+y_2)^2}{b^2}}.$$

5.4.2. Orthogonal tangents and orthoptic set

The orthogonality of the two tangents at P_i $(i = 1, 2)$ of $\mathcal{H}$, means that

$$\tau_1 \cdot \tau_2 = 0 \ \text{ or } \ \nu_1 \cdot \nu_2 = 0 \ \text{ or } \ \frac{x_1x_2}{a^4} + \frac{y_1y_2}{b^4} = 0.$$

To find the intersection $\tilde{P} = (\tilde{x}, \tilde{y})$ of the two tangents, we have to solve the linear system

$$\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Using the orthogonality property of the family of normal vectors, we get

$$\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\nu_1^t}{|\nu_1|^2} & \frac{\nu_2^t}{|\nu_2|^2} \end{bmatrix}$$

and then

$$\tilde{P} = \frac{1}{|\nu_1|^2}\nu_1 + \frac{1}{|\nu_2|^2}\nu_2.$$

Moreover, using the orthogonality condition

$$\tilde{x}^2 + \tilde{y}^2 = \tilde{P} \cdot \tilde{P} = \frac{1}{|\nu_1|^2} + \frac{1}{|\nu_2|^2}.$$

Let us look at the inverse. We have

$$I = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \begin{bmatrix} \frac{\nu_1^t}{|\nu_1|^2} & \frac{\nu_2^t}{|\nu_2|^2} \end{bmatrix},$$

and also

$$\begin{aligned} I &= \begin{bmatrix} \frac{\nu_1^t}{|\nu_1|^2} & \frac{\nu_2^t}{|\nu_2|^2} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \\ &= \frac{1}{|\nu_l|^2}\nu_1^t\nu_1 + \frac{1}{|\nu_2|^2}\nu_2^t\nu_2. \end{aligned}$$

Let us observe that

$$\begin{bmatrix} a^2 & 0 \\ 0 & -b^2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & \iota b \end{bmatrix} I \begin{bmatrix} a & 0 \\ 0 & \iota b \end{bmatrix} = \sum_{i=1}^{2} \frac{1}{|\nu_i|^2} \begin{bmatrix} a & 0 \\ 0 & \iota b \end{bmatrix} \nu_i^t \nu_i \begin{bmatrix} a & 0 \\ 0 & \iota b \end{bmatrix}$$

where $\iota^2 = -1$, and taking the trace on both sides, we get

$$a^2 - b^2 = \sum_{i=1}^{2} \frac{1}{|\nu_i|^2} \left[a^2 \frac{x_i^2}{a^4} - b^2 \frac{y_i^2}{b^4} \right] = \frac{1}{|\nu_1|^2} + \frac{1}{|\nu_2|^2}.$$

Then

$$\tilde{x}^2 + \tilde{y}^2 = a^2 - b^2,$$

so the point $\tilde{P}$ is on the circle of radius $\sqrt{a^2 - b^2}$, under the condition that $a^2 - b^2 \geq 0$.

Let us consider that the aymptotes $y = \pm\frac{b}{a}x$ are two tangents to the hyperbola at infinity. Under the condition that $a > b$, any point on the circle of radius $\sqrt{a^2 - b^2}$ is the intersection of two orthogonal tangents to the hyperbola. To see this fact, first consider a point Q on this circle such that the ray $Q(\lambda) = \lambda Q$ intersect the segment (a, y) with $y \in [-b, b]$ and a $\lambda > 0$. Let $\lambda \in (0, \lambda_0)$ for λ_0 such that $Q(\lambda_0) \in \mathcal{H}$. For any $Q(\lambda)$ we can find two tangents to the branch in $x > 0$ to the hyperbola, and the angle between the tangent increases continuously from a value less that $\pi/2$, because $a > b$, to $+\infty$ for λ increasing in $(0, \lambda_0)$. So, for a point the angle is a right angle, so the two tangents are perpendicular. From the preceding computation this point is on the circle of radius $\sqrt{a^2 - b^2}$. We proceed similarly for a point Q on this circle such that the ray $Q(\lambda) = \lambda Q$ intersect the segment (x, b) with $x \in [-a, a]$ for a $\lambda > 0$. In this case for any $Q(\lambda)$ we can find two tangents to the hyperbola, one on the branch in $x > 0$ and a second on the branch in $x < 0$. The two points of tangency on the hyperbola have a $y < 0$. Now let $\lambda \in (0, +\infty)$, the angle between the tangents decrease continuously from a value larger $\pi/2$, because $a > b$, to 0. So for one point the angle is $\pi/2$, the tangents are perpendiculat and the point is on the circle. For the degenerate case $a = b$, the tangents (at infinity) are the asymptotes an $O = (0, 0)$ is the unique point in the orthoptic set.

In conclusion, if the orthoptic set exists it is the circle of radius $\sqrt{a^2 - b^2}$ also known as Monge's circle.

5.4.3. Parallel rays

We consider two parallel rays F_1P_1 and F_2P_2, with $P_i \in \mathcal{E}$ for $i = 1, 2$, and we would like to find the intersection of the tangents to the hyperbola at those two points P_i. We consider two appropriate sets of directions to scan all the possible directions.

For the first set of directions, let $\eta = (a\cosh(\theta), b\sinh(\theta))$ for $\theta \in \mathbb{R}$. It is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with $x > 0$. We consider a point $F = (\sigma, 0)$ with $|\sigma| > a$, and write

$$P = F + \lambda\eta = (\sigma, 0) + \lambda(a\cosh(\theta), b\sinh(\theta)),$$

and the condition $P \in \mathcal{H}$ leads to

$$\lambda = \lambda_\pm(\sigma) = -\frac{\sigma}{a}\cosh(\theta) \pm \sqrt{\left(\frac{\sigma}{a}\right)^2\cosh^2(\theta) - \left(\left(\frac{\sigma}{a}\right)^2 - 1\right)}.$$

For P_1, associated to the focus F_1, we have $\sigma = c$ and $\lambda_-(c) < \lambda_+(c) < 0$. So, let us set

$$P_1^+ = F_1 + \lambda_+(c)(a\cosh(\theta), b\sinh(\theta)),$$

which is on the branch of $\mathcal{H}$, with $x > 0$, and

$$P_1^- = F_1 + \lambda_-(c)(a\cosh(\theta), b\sinh(\theta)),$$

which is on the branch of $\mathcal{H}$, with $x < 0$.

For P_2, associated to the focus F_2, we have $\sigma = -c$, and $0 < \lambda_-(-c) < \lambda_+(-c)$. So, let us set

$$P_2^+ = F_2 + \lambda_+(-c)(a\cosh(\theta), b\sinh(\theta))$$

which is on the branch of $\mathcal{H}$, with $x > 0$, and

$$P_2^- = F_2 + \lambda_-(-c)(a\cosh(\theta), b\sinh(\theta))$$

which is on the branch of $\mathcal{H}$, with $x < 0$.

We remark that

$$\lambda_+(c) = -\lambda_-(-c) \quad \text{and} \quad \lambda_-(c) = -\lambda_+(-c)$$

and since $F_2 = -F_1$ we have

$$P_1^+ = -P_2^- \quad \text{and} \quad P_1^- = -P_2^+.$$

We then select $P_1 = P_1^+$ and $P_2 = P_2^+$, both on the same branch of $\mathcal{H}$ with $x > 0$. Then we look at the intersection of the tangents to the hyperbola at those points. This intersection is the point

$$\tilde{P} = \frac{a}{\sqrt{c^2\cosh^2(\theta) - b^2}}\left(a\cosh(\theta), b\sinh(\theta)\right),$$

and we observe that

$$\tilde{x}^2 + \tilde{y}^2 = a^2.$$

Moreover we can directly show that $F_1\tilde{P} \cdot \tau_2 = 0$ and $F_2\tilde{P} \cdot \tau_1 = 0$.

For the second set of directions, let $\eta = (a\sinh(\theta), b\cosh(\theta))$ for $\theta \in \mathbb{R}$. It is a point on the hyperbola $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $y > 0$. We consider a point $F = (\sigma, 0)$ with $|\sigma| > a$, and write

$$P = F + \lambda\eta = (\sigma, 0) + \lambda(a\sinh(\theta), b\cosh(\theta)).$$

The condition $P \in \mathcal{H}$ leads to

$$\lambda = \lambda_\pm(\sigma) = \frac{\sigma}{a}\sinh(\theta) \pm \sqrt{\left(\frac{\sigma}{a}\right)^2\sinh^2(\theta) + \left(\left(\frac{\sigma}{a}\right)^2 - 1\right)}.$$

For P_1, associated to the focus F_1, we have $\sigma = c$, and $\lambda_-(c) < 0 < \lambda_+(c)$. So, let us set

$$P_1^+ = F_1 + \lambda_+(c)(a\sinh(\theta), b\cosh(\theta)),$$

and

$$P_1^- = F_1 + \lambda_-(c)(a\sinh(\theta), b\cosh(\theta))$$

which are both on the branch of $\mathcal{H}$ for $x > 0$.

For P_2, associated to the focus F_2, we have $\sigma = -c$ and $\lambda_-(-c) < 0 < \lambda_+(-c)$. So, let us set

$$P_2^+ = F_2 + \lambda_+(-c)(a\sinh(\theta), b\cosh(\theta)),$$

and

$$P_2^- = F_2 + \lambda_-(-c)(a\sinh(\theta), b\cosh(\theta))$$

which are both on the branch of $\mathcal{H}$ for $x < 0$.

We remark that

$$\lambda_+(c) = -\lambda_-(-c) \quad \text{and} \quad \lambda_-(c) = -\lambda_+(-c)$$

and since $F_2 = -F_1$ we have

$$P_1^+ = -P_2^- \quad \text{and} \quad P_1^- = -P_2^+.$$

We then select $P_1 = P_1^+$ and $P_2 = P_2^+$, one on each branche of $\mathcal{H}$ with $y > 0$. Then we look at the intersection of the tangents to the hyperbola at those points. This intersection is the point

$$\tilde{P} = -\frac{a}{\sqrt{c^2\sinh^2(\theta) + b^2}}\left(a\sinh(\theta), b\cosh(\theta)\right),$$

and again we observe that

$$\tilde{x}^2 + \tilde{y}^2 = a^2.$$

Also we can directly show that $F_1\tilde{P}\cdot\tau_2 = 0$ and $F_2\tilde{P}\cdot\tau_1 = 0$.

In both cases, $\tilde{P} = (\tilde{x}, \tilde{y})$ is on the *principal circle* of the hyperbola, the circle of equation $\tilde{x}^2 + \tilde{y}^2 = a^2$.

Those observations lead to a geometric construction of an hyperbola by rotating a rectangle. Draw a rectangle such that two parallel sides pass through the foci with vertices on the circle of radius a centered at $O = (0,0)$. The intersections of the lines passing through the foci and parallel to the diagonals of the rectangle intersect (extended) sides of the rectangle at points on $\mathcal{H}$. So by rotating the rectangle we can find $\mathcal{H}$ pointwise.

5.4.4. Tangents to endpoint of a secant

Using the notation of the first set of directions of the preceding section, we consider

$$\lambda_\pm(\sigma) = -\frac{\sigma}{a}\cosh(\theta) \pm \sqrt{\left(\frac{\sigma}{a}\right)^2\cosh^2(\theta) - \left(\left(\frac{\sigma}{a}\right)^2 - 1\right)}$$

for any $|\sigma| > a$ and $\theta \in \mathbb{R}$. Then set

$$\begin{cases} P_1 &= P_1^+ &= F_1 + \lambda_+(c)(a\cosh(\theta), b\sinh(\theta)) &= (x_1, y_1) \\ P_2 &= P_1^- &= F_1 + \lambda_-(c)(a\cosh(\theta), b\sinh(\theta)) &= (x_2, y_2). \end{cases}$$

In this way, F_1 is on the secant P_1P_2 which connects two points on the two different branches of $\mathcal{H}$. The point of intersection $\tilde{P}$ of the tangents to the ellipse at P_i, $(i = 1, 2)$, is

$$\tilde{P} = (\tilde{x}, \tilde{y}) = \left(a^2/c, ab\coth(\theta)/c\right).$$

Using the notation of the second set of directions of the preceding section, we consider

$$\lambda_\pm(\sigma) = \frac{\sigma}{a}\sinh(\theta) \pm \sqrt{\left(\frac{\sigma}{a}\right)^2\sinh^2(\theta) + \left(\left(\frac{\sigma}{a}\right)^2 - 1\right)}$$

for any $|\sigma| > a$ and $\theta \in \mathbb{R}$. Then set

$$\begin{cases} P_1 = P_1^+ = F_1 + \lambda_+(c)(a\sinh(\theta), b\cosh(\theta)) = (x_1, y_1) \\ P_2 = P_1^- = F_1 + \lambda_-(c)(a\sinh(\theta), b\cosh(\theta)) = (x_2, y_2). \end{cases}$$

In this way, F_1 is on the secant P_1P_2 which connect two points on the same branch of $\mathcal{H}$. The point of intersection $\tilde{P}$ of the tangents to the ellipse at P_i, $(i = 1, 2)$, is

$$\tilde{P} = (\tilde{x}, \tilde{y}) = \left(a^2/c, ab\tanh(\theta)/c\right).$$

Considering both cases, the set of those points $\tilde{P}$ with respect to the focus F_1 is the vertical line $x = a^2/c$ and, by symmetry for the focus F_2, on the vertical line $x = -a^2/c$. These lines are also called the *directrices* of the hyperbola.

6. Conclusion

In this paper, for each conics, namely parabola, ellipse, and hyperbola, we have presented its cartesian equation and considered their tangents. Elementary proofs of their reflexion property are given. Then we have considered orthoptic sets and given elementary, but not wellknown, determination of this set. For ellipse and hyperbola, we have also obtained its principal circle and their directrices both as the intersections of a family of pair of tangents. There are much more to write about conics but we have limited our presentation to intersections of tangents.

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