Asian Journal of Advanced Research and Reports

Volume 17, Issue 8, Page 128-169, 2023; Article no.AJARR.100417
ISSN: 2582-3248

Generalized Horadam-Leonardo Numbers and Polynomials

Yüksel Soykan [a*]

[a] *Department of Mathematics, Faculty of Science, Zonguldak Bülent Ecevit University, Zonguldak-67100, Turkey.*

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/AJARR/2023/v17i8511

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here:
https://www.sdiarticle5.com/review-history/100417

Received: 20/03/2023
Accepted: 26/05/2023
Published: 05/06/2023

Original Research Article

ABSTRACT

In this study, we define and investigate some linear third order polynomials called the generalized Horadam-Leonardo polynomials (with its two special cases, namely), (r, s)-Horadam-Leonardo and (r, s)-Horadam-Leonardo-Lucas polynomials. We give Binet's formulas, generating functions, Simson formulas, and the sum formulas for these polynomial sequences. Also, we present some identities and matrices related to these polynomials. Furthermore, we present some special cases of generalized Horadam-Leonardo polynomials, namely, generalized Leonardo, generalized John, generalized Ernst, generalized Pisano, generalized Edouard and generalized Bigollo numbers.

Keywords: Horadam polynomials; fibonacci polynomials; tribonacci polynomials; horadam-leonardo polynomials; third order recurrence relations.

2020 Mathematics Subject Classification: 11B83, 11B37, 11B39.

**Corresponding author: E-mail: yuksel_soykan@hotmail.com;*

1 INTRODUCTION: HORADAM (GENERALIZED FIBONACCI) POLYNOMIALS

Before defining and investigating the generalized Horadam-Leonardo polynomials we recall the definition and some properties of Horadam polynomials and its two special cases. The generalized Fibonacci polynomials (or Horadam polynomials)

$$\{V_n(V_0(x),V_1(x);r(x),s(x))\}_{n\geq 0}$$

(or $\{V_n(x)\}_{n\geq 0}$ or shortly $\{V_n\}_{n\geq 0}$) is defined by

$$\begin{aligned} V_n(x) &= r(x)V_{n-1}(x)+s(x)V_{n-2}(x), \\ V_0(x) &= a(x), V_1(x)=b(x), \quad n\geq 2 \end{aligned} \tag{1.1}$$

where $V_0(x),V_1(x)$ are arbitrary real (or complex) polynomials with real coefficients and $r(x)$, $s(x)$ are polynomials with real coefficients with $r(x)\neq 0, s(x)\neq 0.$

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative indexes by defining

$$V_{-n}(x)=-\frac{r(x)}{s(x)}V_{-n+1}(x)+\frac{1}{s(x)}V_{-n+2}(x)$$

for $n=1,2,3,...$ when $s(x)\neq 0$. Thus, recurrence (1.1) holds for all integers n. Notice that $V_{-n}(x)$ need not to be a polynomial in the ordinary sense.

We can give some references on special cases of second-order linear recurrence sequences of polynomials and numbers. For example, see [1,2,3,4,5,6,7] for papers and [8,9,10,11,12,13,14] for books.

Binet's formula of generalized Fibonacci polynomials can be given using its characteristic equation which is given as

$$y^2-r(x)y-s(x)=0. \tag{1.2}$$

The roots of characteristic equation are given as

$$\alpha(x):=\alpha=\frac{1}{2}(r(x)+\sqrt{r^2(x)+4s(x)}),\beta(x):=\beta=\frac{1}{2}(r(x)-\sqrt{r^2(x)+4s(x)}), \tag{1.3}$$

and the followings hold

$$\alpha+\beta=r(x),\ \alpha\beta=-s(x).$$

If α and β of characteristic equation (1.2) are distinct, i.e., $\alpha\neq\beta$ then $r^2(x)+4s(x)\neq 0$ and if α and β of characteristic equation (1.2) are equal, i.e., $\alpha=\beta$ then (1.2) can be written as

$$\begin{aligned} y^2-r(x)y-s(x) &= y^2-2\alpha y+\alpha^2 \\ &= (y-\alpha)^2 \\ &= 0 \end{aligned}$$

and, in this case,

$$\alpha=\frac{r(x)}{2},r(x)=2\alpha,s(x)=-\alpha^2=-\frac{r^2(x)}{4},r^2(x)+4s(x)=0.$$

Next, we can define two specific cases of the polynomials $V_n(x)$. $(r(x),s(x))$-Fibonacci polynomials $\{M_n(0,1;r(x),s(x))\}_{n\geq 0}$ (or shortly, $M_n(x)$) and $(r(x),s(x))$-Lucas polynomials $\{N_n(2,r(x);r(x),s(x))\}_{n\geq 0}$ (or shortly, $N_n(x)$) are defined by the second-order recurrence relations

$$M_{n+2}(x) = r(x)M_{n+1}+s(x)M_n(x), \quad M_0(x)=0,M_1(x)=1, \tag{1.4}$$

$$N_{n+2}(x) = r(x)N_{n+1}+s(x)N_n(x), \quad N_0(x)=2,N_1(x)=r(x), \tag{1.5}$$

respectively

$\{M_n(x)\}_{n\geq 0}$ and $\{N_n(x)\}_{n\geq 0}$ can be extended to negative indexes by defining

$$\begin{aligned} M_{-n}(x) &= -\frac{r(x)}{s(x)}M_{-(n-1)}(x)+\frac{1}{s(x)}M_{-(n-2)}(x), \\ N_{-n}(x) &= -\frac{r(x)}{s(x)}N_{-(n-1)}(x)+\frac{1}{s(x)}N_{-(n-2)}(x), \end{aligned}$$

for $n=1,2,3,...$ respectively. Thus, recurrences (1.4) and (1.5) hold for all integers n.

NOTE: Throughout the rest of the paper, we use, respectively,

$$V_n, r, s, V_0, V_1, \alpha, \beta, M_n, N_n, M_0, M_1, N_0, N_1$$

instead of

$$V_n(x), r(x), s(x), V_0(x), V_1(x), \alpha(x), \beta(x), M_n(x), N_n(x), M_0(x), M_1(x), N_0(x), N_1(x).$$

For instance, we write

$$V_n = rV_{n-1}+sV_{n-2}, \quad V_0=a, V_1=b, \ \ n\geq 2$$

for the equation (1.1).

Using α,β and recurrence relation (1.1), Binet's formula of V_n can be given as follows:

Theorem 1.1.

(a) *($\alpha\neq\beta$: Distinct Roots Case) Binet's formula of generalized Fibonacci polynomials is*

$$V_n=\frac{r_1\alpha^n}{\alpha-\beta}+\frac{r_2\beta^n}{\beta-\alpha}=\frac{r_1\alpha^n-r_2\beta^n}{\alpha-\beta} \tag{1.6}$$

where

$$r_1=V_1-\beta V_0, \ r_2=V_1-\alpha V_0.$$

(b) *($\alpha=\beta$: Single Root Case) Binet's formula of generalized Fibonacci polynomials is*

$$V_n=(D_2n+D_1)\alpha^n \tag{1.7}$$

where

$$\begin{aligned} D_1 &= V_0, \\ D_2 &= \frac{1}{\alpha}\left(V_1-\alpha V_0\right). \end{aligned}$$

Note that Binet's formulas of M_n and N_n can be given, respectively, as follows:

$$M_n=\left\{\begin{array}{ccc} \dfrac{\alpha^n-\beta^n}{\alpha-\beta} & , & \text{if (Distinct Roots Case): } \alpha\neq\beta \\ n\alpha^{n-1} & , & \text{if (Single Root Case): } \alpha=\beta \end{array}\right. ,$$

$$N_n=\left\{\begin{array}{ccc} \alpha^n+\beta^n & , & \text{if (Distinct Roots Case): } \alpha\neq\beta \\ 2\alpha^n & , & \text{if (Single Root Case): } \alpha=\beta \end{array}\right. .$$

Next, we define two sequences related to (r,s)-Fibonacci polynomials and (r,s)-Fibonacci-Lucas polynomials. For $r,\ s$ satisfying Eq. (1.4)-(1.5), (r,s)-Horadam-Leonardo polynomials and (r,s)-Horadam-Leonardo-Lucas polynomials are defined as

$$G_n(x)=rG_{n-1}(x)+sG_{n-2}(x)+1 \ \text{ with } \ G_0(x)=0, G_1(x)=1, \ n\geq 2, \tag{1.8}$$

and

$$H_n(x)=rH_{n-1}(x)+sH_{n-2}(x)+(1-s-r) \ \text{ with } H_0(x)=3, H_1(x)=r+1, \ n\geq 2, \tag{1.9}$$

respectively.

Note that $G_2(x) = r + 1$ and $H_2(x) = r^2 + 2s + 1$. We can present the first few values of Horadam-Leonardo polynomials and Horadam-Leonardo-Lucas polynomials as

$$0, 1, r + 1, r^2 + r + s + 1, r^3 + r^2 + r + 2rs + s + 1, ...$$

and

$$3, r + 1, r^2 + 2s + 1, r^3 + 3sr + 1, r^4 + 4r^2 s + 2s^2 + 1, ...$$

respectively. Note also that from the equations (1.8) and (1.9), we get

$$\begin{aligned} sG_{n-3}(x) &= G_{n-1}(x) - rG_{n-2}(x) - 1, \\ sH_{n-3}(x) &= H_{n-1}(x) - rH_{n-2}(x) - (1 - s - r), \end{aligned}$$

and so the sequences $\{G_n(x)\}$ and $\{H_n(x)\}$ satisfy third order linear recurrences and given as

$$G_n(x) = (r + 1)G_{n-1}(x) + (s - r)G_{n-2}(x) - sG_{n-3}(x), \quad (1.10)$$
$$H_n(x) = (r + 1)H_{n-1}(x) + (s - r)H_{n-2}(x) - sH_{n-3}(x). \quad (1.11)$$

Remark. *Note that if $1 - s - r = 0$, i.e., $s = 1 - r$,$r = 1 - s$, then we see from (1.9) and (1.11) that the sequence $\{H_n(x)\}$ both have second order and third order linear relations. In this case, we get*

$$H_2(x) = r^2 + 2s + 1 = r^2 - 2r + 3 = s^2 + 2.$$

In this paper, we also consider and investigate the case $1 - s - r = 0$ by considering it in the general case, i.e. in (1.9) and (1.11) that is (2.3).

Note that if we define a sequence of polynomials as

$$Y_n(x) = rY_{n-1}(x) + sY_{n-2}(x) + c(x), \quad \text{with} \quad Y_0(x) = d_1(x), Y_1(x) = d_2(x), \quad n \geq 2$$

where r, s satisfying (1.1) and $Y_0(x), Y_1(x)$ are arbitrary real (or complex) polynomials with real coefficients and $c(x)$ is a polynomial with real coefficients, then since

$$sY_{n-3}(x) = Y_{n-1}(x) - rY_{n-2}(x) - c(x),$$

we get

$$Y_n(x) = (r + 1)Y_{n-1}(x) + (s - r)Y_{n-2}(x) - sY_{n-3}(x).$$

2 GENERALIZED HORADAM-LEONARDO POLYNOMIALS

In this section, for r, s satisfying (1.1), we present and investigate a new sequence and its two special cases, namely the generalized Horadam-Leonardo, (r, s)-Horadam-Leonardo and (r, s)-Horadam-Leonardo-Lucas polynomials.

For r, s satisfying (1.1), generalized Horadam-Leonardo polynomials $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2; r + 1, s - r, -s)\}_{n\geq 0}$ (or, shortly, $\{W_n(x)\}_{n\geq 0}$) is defined by (the third-order recurrence relation)

$$W_n(x) = (r + 1)W_{n-1}(x) + (s - r)W_{n-2}(x) - sW_{n-3}(x) \quad (2.1)$$

with initial values $W_0(x) = c_0(x), W_1(x) = c_1(x), W_2(x) = c_2(x)$ not all being zero and $W_0(x), W_1(x), W_2(x)$ are arbitrary real (or complex) polynomials with real coefficients.

$\{W_n(x)\}_{n\geq 0}$ can be extended to negative indexes by defining

$$W_{-n}(x) = \frac{s - r}{s} W_{-(n-1)}(x) + \frac{r + 1}{s} W_{-(n-2)}(x) - \frac{1}{s} W_{-(n-3)}(x)$$

for $n = 1, 2, 3, ...$ Thus, recurrence (2.1) holds for all integer $n.$ Notice that $W_{-n}(x)$ need not to be a polynomial in the ordinary sense for $n \geq 1.$

Generalized Horadam-Leonardo polynomial are special cases of generalized Tribonacci polynomials. See [15,16,17,18] for some references on generalized Tribonacci polynomials and its special cases.

Note that the sequences $\{G_n(x)\}$ and $\{H_n(x)\}$, defined in the section Introduction, are the special cases of the generalized Horadam-Leonardo polynomials $\{W_n(x)\}$. For the benefit, we give the definition of these two special cases of the sequence $\{W_n(x)\}$ in this section too. (r, s)-Horadam-Leonardo and (r, s)-Horadam-Leonardo-Lucas polynomials are defined, by (the third-order recurrence relations)

$$\begin{aligned} G_n(x) &= (r+1)G_{n-1}(x) + (s-r)G_{n-2}(x) - sG_{n-3}(x), \\ G_0(x) &= 0, G_1(x) = 1, G_2(x) = r+1, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} H_n(x) &= (r+1)H_{n-1}(x) + (s-r)H_{n-2}(x) - sH_{n-3}(x), \\ H_0(x) &= 3, H_1(x) = r+1, H_2(x) = r^2 + 2s + 1, \end{aligned} \tag{2.3}$$

respectively. The sequences $\{G_n(0, 1, r+1; r+1, s-r, -s)\}_{n\geq 0}$ and $\{H_n(3, r+1, r^2+2s+1; r+1, s-r, -s)\}_{n\geq 0}$ can be extended to negative indexes by defining

$$\begin{aligned} G_{-n}(x) &= \frac{s-r}{s}G_{-(n-1)}(x) + \frac{r+1}{s}G_{-(n-2)}(x) - \frac{1}{s}G_{-(n-3)}(x), \\ H_{-n}(x) &= \frac{s-r}{s}H_{-(n-1)}(x) + \frac{r+1}{s}H_{-(n-2)}(x) - \frac{1}{s}H_{-(n-3)}(x), \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively.

Remark. *Throughout the rest of the paper, we use, respectively,*

$$W_n, W_0, W_1, W_2, G_n, G_0, G_1, G_2, H_n, H_0, H_1, H_2,$$

instead of

$$W_n(x), W_0(x), W_1(x), W_2(x), G(x), G_0(x), G_1(x), G_2(x), H(x), H_0(x), H_1(x), H_2(x)$$

unless otherwise stated. For instance, we write

$$W_n = (r+1)W_{n-1} + (s-r)W_{n-2} - sW_{n-3}, \ W_0 = c_0, W_1 = c_1, W_2 = c_2, \ n \geq 3$$

for the equation (2.1). Also we write K_n, K_0, K_1, K_2 instead of $K_n(x)$ with initial conditions $K_0(x), K_1(x), K_2(x)$ for any subsequence $\{K_n(x)\}$ of $\{W_n\}$.

When r, s, W_0, W_1, W_2 are real numbers we call generalized Horadam-Leonardo, (r, s)-Horadam-Leonardo and (r, s)-Horadam-Leonardo-Lucas polynomials as generalized Horadam-Leonardo, (r, s)-Horadam-Leonardo and (r, s)-Horadam-Leonardo-Lucas numbers (sequences).

Now, we give a few special cases of generalized Horadam-Leonardo sequence are given as follows (Table 1):

Table 1. Several cases of generalized Horadam-Leonardo sequence

No	Numbers (Sequences)	r, s	Notation	References
1	Generalized Leonardo	$r = 1, s = 1$	$\{W_n(W_0, W_1, W_2; 2, 0, -1)\}$	[19]
2	Generalized John	$r = 2, s = 1$	$\{W_n(W_0, W_1, W_2; 3, -1, -1)\}$	[20]
3	Generalized Ernst	$r = 1, s = 2$	$\{W_n(W_0, W_1, W_2; 2, 1, -2)\}$	[21]
4	Generalized Pisano	$r = 1, s = -\frac{1}{4}$	$\{W_n(W_0, W_1, W_2; 2, -\frac{5}{4}, \frac{1}{4})\}$	[22]
5	Generalized Edouard	$r = 6, s = -1$	$\{W_n(W_0, W_1, W_2; 7, -7, 1)\}$	[23]
6	Generalized Bigollo	$r = 3, s = -2$	$\{W_n(W_0, W_1, W_2; 4, -5, 2)\}$	[24]

Next, we give several values of the (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials with positive and negative indexes (Table 2):

Table 2. Several special third-order numbers with positive and negative indexes.

n	0	1	2	3	4
G_n	0	1	$r+1$	$r^2+r+s+1$	$r^3+r^2+r+2rs+s+1$
G_{-n}		0	$-\frac{1}{s}$	$\frac{1}{s^2}(r-s)$	$-\frac{1}{s^3}\left(s-rs+r^2+s^2\right)$
H_n	3	$r+1$	r^2+2s+1	$r^3+3sr+1$	$r^4+4r^2s+2s^2+1$
H_{-n}		$-\frac{1}{s}(r-s)$	$\frac{1}{s^2}\left(r^2+s^2+2s\right)$	$-\frac{1}{s^3}\left(r^3+3rs-s^3\right)$	$\frac{1}{s^4}\left(r^4+4r^2s+s^4+2s^2\right)$

Several special cases of (r,s)-Horadam-Leonardo sequence $\{G_n(0,1,r+1;r+1,s-r,-s)\}$ and (r,s)-Horadam-Leonardo-Lucas sequence $\{H_n(3,r+1,r^2+2s+1;r+1,s-r,-s)\}$ are given as follows:

1. $G_n(0,1,2;2,0,-1)=G_n$, modified Leonardo sequence, see [19].
2. $H_n(3,2,4;2,0,-1)=H_n$, Leonardo-Lucas sequence, see [19].
3. $G_n(0,1,3;3,-1,-1)=J_n$, John sequence, see [20].
4. $H_n(3,3,7;3,-1,-1)=H_n$, John-Lucas sequence, see [20].
5. $G_n(0,1,2;2,1,-2)=E_n$, Ernst sequence, see [21].
6. $H_n(3,2,6;2,1,-2)=H_n$, Ernst-Lucas sequence, see [21].
7. $G_n(0,1,2;2,-\frac{5}{4},\frac{1}{4})=P_n$, Pisano sequence, see [22].
8. $H_n(3,2,\frac{3}{2};2,-\frac{5}{4},\frac{1}{4})=R_n$, Pisano-Lucas sequence, see [22].
9. $G_n(0,1,7;7,-7,1)=E_n$, Edouard sequence, see [23].
10. $H_n(3,7,35;7;7,-7,1)=K_n$, Edouard-Lucas sequence, see [23].
11. $G_n(0,1,4;4,-5,2)=B_n$, Bigollo sequence, see [24].
12. $H_n(3,4,6;1,4,-5,2)=C_n$, Bigollo-Lucas sequence, see [24].

The characteristic equation of W_n is given by

$$y^3-(r+1)y^2-(s-r)y+s=(y^2-ry-s)(y-1)=0. \tag{2.4}$$

and the roots of characteristic equation are given as

$$\alpha=\frac{r+\sqrt{r^2+4s}}{2},\ \ \beta=\frac{r-\sqrt{r^2+4s}}{2},\ \gamma=1,$$

where α and β are as in (1.3).

Next, we give Binet's formula of generalized Horadam-Leonardo polynomials.

Corollary 2.1. *According to the roots of characteristic equation (2.4), Binet's formula of generalized Horadam-Leonardo polynomials is given by:*

(a) *($\alpha\neq\beta\neq\gamma=1$, Three Distinct Roots Case)*

$$\begin{aligned} W_n &= \frac{W_2-(\beta+1)W_1+\beta W_0}{(\alpha-\beta)(\alpha-1)}\alpha^n+\frac{W_2-(\alpha+1)W_1+\alpha W_0}{(\beta-\alpha)(\beta-1)}\beta^n+\frac{W_2+(-(r+1)+1)W_1+(-s)W_0}{(-s)-(r+1)+2} \\ &= \frac{(\alpha W_2+\alpha(-(r+1)+\alpha)W_1+(-s)W_0)}{(r+1)\alpha^2+2((s-r)-r)\alpha+3(-s)}\alpha^n+\frac{(\beta W_2+\beta(-(r+1)+\beta)W_1+(-s)W_0)}{(r+1)\beta^2+2((s-r)-r)\beta+3(-s)}\beta^n \\ &\quad +\frac{W_2+(-(r+1)+1)W_1+(-s)W_0}{(r+1)+2((s-r)-r)+3(-s)}. \end{aligned}$$

(b) *($\alpha \neq \beta = \gamma = 1$, Two Distinct Roots Case)*

$W_n = \frac{1}{(1-(-s))^2}((W_2 - 2W_1 + W_0)\alpha^n + (-W_2 + 2W_1 + (-s)((-s)-2)W_0) + (1-(-s))(W_2 - (1+(-s))W_1 + (-s)W_0)n).$

(c) *($\alpha = \beta = \gamma = 1 = \frac{(r+1)}{3}$, Single Root Case)*

$$W_n = \frac{1}{2}(n(n-1)W_2 - 2n(n-2)W_1 + (n-1)(n-2)W_0).$$

Proof. In [25, Corollary 7.], replace r, s and t with $r+1, s-r, -s$, respectively. □

Now, we present the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

Lemma 2.2. *Assume that $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized Horadam-Leonardo polynomials $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n z^n$ is given as*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - (r+1)W_0)z + (W_2 - (r+1)W_1 - (s-r)W_0)z^2}{1-(r+1)z-(s-r)z^2-(-s)z^3}.$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Lemma 9.]. □

As particular examples (generating functions of (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials), from the last Lemma, we get

Corollary 2.3. *The generating functions of (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials are*

$$\begin{aligned}\sum_{n=0}^{\infty} G_n z^n &= \frac{z}{1-(r+1)z-(s-r)z^2-(-s)z^3},\\ \sum_{n=0}^{\infty} H_n z^n &= \frac{3-2(r+1)z-(s-r)z^2}{1-(r+1)z-(s-r)z^2-(-s)z^3},\end{aligned}$$

respectively.

Proof. In the last Lemma, put $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = (r+1)$ and take $W_n = H_n$ with $H_0 = 3, H_1 = (r+1), H_2 = r^2 + 2s + 1$, respectively. □

Next, as a special case of Corollary 2.1, we present Binet formulas of (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials.

Corollary 2.4. *Binet's formulas of (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials, for all integers n, are given as follows*

(a) *($\alpha \neq \beta \neq \gamma = 1$, Three Distinct Roots Case)*

$$\begin{aligned}G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)} + \frac{1}{(1-\alpha)(1-\beta)}\\ &= \frac{\alpha^{n+2}}{(r+1)\alpha^2+2(s-r)\alpha+3(-s)} + \frac{\beta^{n+2}}{(r+1)\beta^2+2(s-r)\beta+3(-s)} + \frac{1}{(r+1)+2(s-r)+3(-s)},\\ H_n &= \alpha^n + \beta^n + 1.\end{aligned}$$

(b) *($\alpha \neq \beta = \gamma = 1$, Two Distinct Roots Case)*

$$G_n = \frac{\alpha^{n+1} + ((1-\alpha)n - \alpha)}{(1-\alpha)^2}, \ H_n = \alpha^n + 2.$$

(c) *($\alpha = \beta = \gamma = 1 = \frac{(r+1)}{3}$, Single Root Case)*

$$G_n = \frac{n(n+1)}{2}, \ H_n = 3.$$

Proof. In Corollary 2.1, put $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = (r+1)$ and take $W_n = H_n$ with $H_0 = 3, H_1 = (r+1), H_2 = r^2 + 2s + 1$, respectively. □

3 SOME SPECIFIC CASES OF GENERALIZED HORADAM-LEONARDO POLYNOMIALS

In this section, we give literature review and present on some special cases of generalized Horadam-Leonardo polynomials.

3.1 Generalized Leonardo Sequence

In this subsection, it is considered that the special case $r = 1, s = 1$. In this case, $r + 1 = 2, s - r = 0, -s = -1$. A generalized Leonardo sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by (the third-order recurrence relations)

$$W_n = 2W_{n-1} - W_{n-3} \tag{3.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$. Next, define three special cases of the sequence $\{W_n\}$. Modified Leonardo sequence $\{G_n\}_{n\geq 0}$, Leonardo-Lucas sequence $\{H_n\}_{n\geq 0}$ and Leonardo sequence $\{l_n\}_{n\geq 0}$ are defined, (by the third-order recurrence relations)

$$\begin{aligned} G_n &= 2G_{n-1} - G_{n-3}, && (3.2) \\ G_0 &= 0, G_1 = 1, G_2 = 2, \\ H_n &= 2H_{n-1} - H_{n-3}, && (3.3) \\ H_0 &= 3, H_1 = 2, H_2 = 4, \\ l_n &= 2l_{n-1} - l_{n-3}, && (3.4) \\ l_0 &= 1, l_1 = 1, l_2 = 3, \end{aligned}$$

respectively. Modified Leonardo, Leonardo-Lucas, Leonardo numbers and Fibonacci, Lucas numbers satisfy the following interrelations:

$$G_n = F_{n+2} - 1, H_n = L_n + 1, l_n = 2F_{n+1} - 1,$$

and

$$5G_n = 3L_{n+1} + L_n - 5, H_n = 2F_{n+1} - F_n + 1, 5l_n = 2L_{n+1} + 4L_n - 5.$$

where the sequence of Fibonacci numbers $\{F_n\}$ and the sequence of Lucas numbers $\{L_n\}$ are defined, respectively, as

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2}, \ \ n \geq 2, \\ F_0 &= 0, \ F_1 = 1, \end{aligned}$$

and

$$\begin{aligned} L_n &= L_{n-1} + L_{n-2}, \;\; n \geq 2, \\ L_0 &= 2, \; L_1 = 1, \end{aligned}$$

For more information on generalized Lenonardo number, see [19].

3.2 Generalized John Sequence

In this subsection, it is considered that the special case $r = 2, s = 1$. In this case, $r+1 = 3, s-r = -1, -s = -1$. A generalized John sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 3W_{n-1} - W_{n-2} - W_{n-3} \tag{3.5}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$. Next, define two special cases of the sequence $\{W_n\}$. John sequence $\{J_n\}_{n\geq 0}$ and John-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined by (the third-order recurrence relations)

$$\begin{aligned} J_n &= 3J_{n-1} - J_{n-2} - J_{n-3}, & (3.6) \\ J_0 &= 0, J_1 = 1, J_2 = 3, \\ H_n &= 3H_{n-1} - H_{n-2} - H_{n-3}, & (3.7) \\ H_0 &= 3, H_1 = 3, H_2 = 7, \end{aligned}$$

respectively. John and John-Lucas and Pell, Pell-Lucas numbers satisfy the following interrelations:

$$J_n = \frac{1}{2}(P_{n+2} - P_{n+1} - 1), H_n = Q_n + 1,$$

and

$$J_n = \frac{1}{4}(Q_{n+1} - 2), H_n = 2P_{n+1} - 2P_n + 1,$$

where Pell sequence $\{P_n\}_{n\geq 0}$ and Pell-Lucas sequence $\{Q_n\}_{n\geq 0}$ are defined, respectively, by (the second-order recurrence relations)

$$\begin{aligned} P_n &= 2P_{n-1} + P_{n-2}, & (3.8) \\ P_0 &= 0, P_1 = 1 \end{aligned}$$

and

$$\begin{aligned} Q_n &= 2Q_{n-1} + Q_{n-2}, & (3.9) \\ Q_0 &= 2, Q_1 = 2. \end{aligned}$$

See [20] for more information on generalized John numbers.

3.3 Generalized Ernst Sequence

In this subsection, it is considered that the special case $r = 1, s = 2$. In this case, $r+1 = 2, s-r = 1, -s = -2$. A generalized Ernst sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relation

$$W_n = 2W_{n-1} + W_{n-2} - 2W_{n-3} \tag{3.10}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$. Next, define two special cases of the sequence $\{W_n\}$. Ernst sequence $\{E_n\}_{n\geq 0}$ and Ernst-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined by (the third-order recurrence relations)

$$\begin{aligned} E_n &= 2E_{n-1} + E_{n-2} - 2E_{n-3}, & (3.11) \\ E_0 &= 0, E_1 = 1, \; E_3 = 2, \\ H_n &= 2H_{n-1} + H_{n-2} - 2H_{n-3}, & (3.12) \\ H_0 &= 3, H_1 = 2, \; H_0 = 6, \end{aligned}$$

respectively. Ernst and Ernst-Lucas and Jacobsthal, Jacobsthal-Lucas numbers satisfy the following interrelations:

$$E_n = \frac{1}{2}(J_{n+2} - 1), H_n = j_n + 1,$$

and

$$18E_n = 5j_{n+1} + 2j_n - 9, 2H_n = 4J_{n+1} - 2J_n + 2.$$

where Jacobsthal sequence $\{J_n\}_{n\geq 0}$ and Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ are defined by (the second-order recurrence relations)

$$\begin{aligned} J_n &= J_{n-1} + 2J_{n-2}, \\ J_0 &= 0, J_1 = 1 \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} j_n &= j_{n-1} + 2j_{n-2}, \\ j_0 &= 2, j_1 = 1, \end{aligned} \tag{3.14}$$

respectively. See [21] for more information on generalized Ernst numbers.

3.4 Generalized Pisano Sequence

In this subsection, it is considered that the special case $r = 1, s = -\frac{1}{4}$. In this case, $r + 1 = 2, s - r = -\frac{5}{4}, -s = \frac{1}{4}$. A generalized Pisano sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by (the third-order recurrence relation)

$$W_n = 2W_{n-1} - \frac{5}{4}W_{n-2} + \frac{1}{4}W_{n-3} \tag{3.15}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$. Next, define two special cases of the sequence $\{W_n\}$. Pisano sequence $\{P_n\}_{n\geq 0}$ and Pisano–Lucas sequence $\{R_n\}_{n\geq 0}$ are defined by (the third-order recurrence relations)

$$\begin{aligned} P_n &= 2P_{n-1} - \frac{5}{4}P_{n-2} + \frac{1}{4}P_{n-3}, && (3.16) \\ P_0 &= 0, P_1 = 1, P_2 = 2, \\ R_n &= 2R_{n-1} - \frac{5}{4}R_{n-2} + \frac{1}{4}R_{n-3}, && (3.17) \\ R_0 &= 3, R_1 = 2, R_2 = \frac{3}{2}, \end{aligned}$$

respectively. Pisano and Pisano–Lucas and modified Oresme, Oresme–Lucas, Oresme numbers satisfy the following interrelations:

$$nP_n = -(n+2)G_n + 4n, P_n = -(n+2)H_n + 4, nP_n = -2(n+2)O_n + 4n,$$

and

$$nR_n = G_n + n, R_n = H_n + 1, nR_n = 2O_n + n,$$

where modified Oresme sequence $\{G_n\}_{n\geq 0}$, Oresme–Lucas sequence $\{H_n\}_{n\geq 0}$ and Oresme sequence $\{O_n\}_{n\geq 0}$ are defined, respectively, by (the second-order recurrence relations)

$$\begin{aligned} G_{n+2} &= G_{n+1} - \frac{1}{4}G_n, \quad G_0 = 0, G_1 = 1, && (3.18) \\ H_{n+2} &= H_{n+1} - \frac{1}{4}H_n, \quad H_0 = 2, H_1 = 1, && (3.19) \\ O_{n+2} &= O_{n+1} - \frac{1}{4}O_n, \quad O_0 = 0, O_1 = \frac{1}{2}. && (3.20) \end{aligned}$$

See [22] for more information on generalized Pisano numbers.

3.5 Generalized Edouard Sequence

In this subsection, it is considered that the special case $r = 6, s = -1$. In this case, $r + 1 = 7, s - r = -7, -s = 1$. A generalized Edouard sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by (the third-order recurrence relations)

$$W_n = 7W_{n-1} - 7W_{n-2} + W_{n-3} \tag{3.21}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$. Next, define two special cases of the sequence $\{W_n\}$. Edouard sequence $\{E_n\}_{n\geq 0}$ and Edouard-Lucas sequence $\{K_n\}_{n\geq 0}$ are defined by (the third-order recurrence relations)

$$\begin{aligned} E_n &= 7E_{n-1} - 7E_{n-2} + E_{n-3}, & (3.22)\\ E_0 &= 0, E_1 = 1, E_2 = 7, \\ K_n &= 7K_{n-1} - 7K_{n-2} + K_{n-3}, & (3.23)\\ K_0 &= 3, K_1 = 7, K_2 = 35. \end{aligned}$$

respectively. Edouard, Edouard-Lucas and balancing, modified Lucas-balancing, Lucas-balancing numbers satisfy the following interrelations:

$$E_n = \frac{1}{4}(B_{n+1} - B_n - 1), K_n = H_n + 1 = 2C_n + 1$$

and

$$32E_n = H_{n+1} + H_n - 8 = 2C_{n+1} + 2C_n - 8, K_n = 2B_{n+1} - 6B_n + 1,$$

where balancing sequence $\{B_n\}_{n\geq 0}$, modified Lucas-balancing sequence $\{H_n\}_{n\geq 0}$ and Lucas-balancing sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by (the second-order recurrence relations)

$$\begin{aligned} B_n &= 6B_{n-1} - B_{n-2}, & (3.24)\\ B_0 &= 0, B_1 = 1, \\ H_n &= 6H_{n-1} - H_{n-2}, & (3.25)\\ H_0 &= 2, H_1 = 6, \\ C_n &= 6C_{n-1} - C_{n-2}, & (3.26)\\ C_0 &= 1, C_1 = 3. \end{aligned}$$

See [23] for more information on generalized Edouard numbers.

3.6 Generalized Bigollo Sequence

In this subsection, it is considered that the special case $r = 3, s = -2$. In this case, $r + 1 = 4, s - r = -5, -s = 2$. A generalized Bigollo sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by (the third-order recurrence relations)

$$W_n = 4W_{n-1} - 5W_{n-2} + 2W_{n-3} \tag{3.27}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$. Next, define two special cases of the sequence $\{W_n\}$. Bigollo sequence $\{B_n\}_{n\geq 0}$ and Bigollo-Lucas sequence $\{C_n\}_{n\geq 0}$ are defined by (the third-order recurrence relations)

$$\begin{aligned} B_n &= 4B_{n-1} - 5B_{n-2} + 2B_{n-3}, \\ B_0 &= 0, B_1 = 1, B_2 = 4, & (3.28)\\ C_n &= 4C_{n-1} - 5C_{n-2} + 2C_{n-3}, \\ C_0 &= 3, C_1 = 4, C_2 = 6. & (3.29) \end{aligned}$$

respectively. Bigollo and Bigollo-Lucas and Mersenne, Mersenne-Lucas numbers satisfy the following interrelations:

$$B_n = 2M_n - n, C_n = H_n + 1,$$

and

$$B_n = 4H_{n+1} - 6H_n - n, 2C_n = 2M_{n+1} - 2M_n + 4$$

where Mersenne sequence $\{M_n\}_{n\geq 0}$ and Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined by (the second-order recurrence relations)

$$\begin{aligned} M_n &= 3M_{n-1} - 2M_{n-2}, & (3.30)\\ M_0 &= 0, M_1 = 1, \\ H_n &= 3H_{n-1} - 2H_{n-2}, & (3.31)\\ H_0 &= 2, H_1 = 3, \end{aligned}$$

respectively. See [24] for more information on generalized Bigollo numbers.

4 SIMSON FORMULAS OF HORADAM-LEONARDO POLYNOMIALS

Next we present Simson's formula of the generalized Horadam-Leonardo polynomials $\{W_n\}$.

Theorem 4.1. *For all integers n, we get*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = (-s)^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}. \tag{4.1}$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Theorem 33.]. □

The previous theorem gives the following results as particular examples.

Corollary 4.2. *For all integers n, Simson's formula of (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials are given by*

$$\begin{vmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{vmatrix} = -(-s)^{n-1}, \quad \begin{vmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{vmatrix} = (r^2+4s)(r+s-1)^2(-s)^{n-2}$$

respectively.

Notice that for all integers n, m, (4.1) can be written as

$$\begin{vmatrix} W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} \end{vmatrix} = (-s)^{n+m} \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}.$$

Next we define

$$\begin{aligned} \Lambda_W(n) = {} & W_{n+2}^3 - r(r-s+1)W_{n+1}^3 + (-s)^2W_n^3 - 2(r+1)W_{n+1}W_{n+2}^2 - (s-r)W_nW_{n+2}^2 \\ & +(r^2+3r-s+1)W_{n+2}W_{n+1}^2 + ((r+1)(-s)+(s-r)^2)W_nW_{n+1}^2 \\ & +(r+1)(-s)W_n^2W_{n+2} + 2(s-r)(-s)W_n^2W_{n+1} - (r^2+r-4s-rs)W_{n+2}W_{n+1}W_n \end{aligned}$$

Then

$$\begin{aligned} \Lambda_W(0) = {} & W_2^3 - r(r-s+1)W_1^3 + (-s)^2W_0^3 - 2(r+1)W_1W_2^2 - (s-r)W_0W_2^2 \\ & +(r^2+3r-s+1)W_2W_1^2 + ((r+1)(-s)+(s-r)^2)W_0W_1^2 \\ & +(r+1)(-s)W_0^2W_2 + 2(s-r)(-s)W_0^2W_1 - (r^2+r-4s-rs)W_2W_1W_0. \end{aligned} \tag{4.2}$$

Notice that Simson's formulas of W_n, G_n H_n can be written in the following forms:

Lemma 4.3. *For all integers n, we get*

(a) $\Lambda_W(n) = (-s)^n \Lambda_W(0)$, *that is*

$$\begin{aligned} & W_{n+2}^3 - r(r-s+1)W_{n+1}^3 + (-s)^2 W_n^3 - 2(r+1)W_{n+1}W_{n+2}^2 - (s-r)W_n W_{n+2}^2 \\ & +(r^2+3r-s+1)W_{n+2}W_{n+1}^2 + ((r+1)(-s)+(s-r)^2)W_n W_{n+1}^2 \\ & +(r+1)(-s)W_n^2 W_{n+2} + 2(s-r)(-s)W_n^2 W_{n+1} - (r^2+r-4s-rs)W_{n+2}W_{n+1}W_n \\ = \; & (-s)^n (W_2^3 - r(r-s+1)W_1^3 + (-s)^2 W_0^3 - 2(r+1)W_1 W_2^2 - (s-r)W_0 W_2^2 \\ & +(r^2+3r-s+1)W_2 W_1^2 + (r^2+s^2-s-3rs)W_0 W_1^2 + (r+1)(-s)W_0^2 W_2 \\ & +2(s-r)(-s)W_0^2 W_1 - (r^2+r-4s-rs)W_2 W_1 W_0). \end{aligned}$$

(b) $\Lambda_G(n) = (-s)^n \Lambda_G(0) = (-s)^{n+1}$, *that is*

$$\begin{aligned} & G_{n+2}^3 - r(r-s+1)G_{n+1}^3 + (-s)^2 G_n^3 - 2(r+1)G_{n+1}G_{n+2}^2 - (s-r)G_n G_{n+2}^2 \\ & +(r^2+3r-s+1)G_{n+2}G_{n+1}^2 + (r^2+s^2-s-3rs)G_n G_{n+1}^2 \\ & +(r+1)(-s)G_n^2 G_{n+2} + 2(s-r)(-s)G_n^2 G_{n+1} - (r^2+r-4s-rs)G_{n+2}G_{n+1}G_n \\ = \; & (-s)^{n+1}. \end{aligned}$$

(c) $\Lambda_H(n) = (-s)^n \Lambda_H(0) = -(r^2+4s)(r+s-1)^2(-s)^n$, *that is*

$$\begin{aligned} & H_{n+2}^3 - r(r-s+1)H_{n+1}^3 + (-s)^2 H_n^3 - 2(r+1)H_{n+1}H_{n+2}^2 - (s-r)H_n H_{n+2}^2 \\ & +((r+1)^2-(s-r))H_{n+2}H_{n+1}^2 + ((r+1)(-s)+(s-r)^2)H_n H_{n+1}^2 \\ & +(r+1)(-s)H_n^2 H_{n+2} + 2(s-r)(-s)H_n^2 H_{n+1} - (r^2+r-4s-rs)H_{n+2}H_{n+1}H_n \\ = \; & -(r^2+4s)(r+s-1)^2(-s)^n. \end{aligned}$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Lemma 35.]. □

5 IDENTITIES OF GENERALIZED HORADAM-LEONARDO POLYNOMIALS

In this section, we get some identities of (r, s)-Horadam-Leonardo and (r, s)-Horadam-Leonardo-Lucas polynomials. First, we present a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 5.1. *The following equalities are true:*

(a) $(-s)^3 H_n = (3rs + r^3 - s^3)G_{n+4} - (4r^2 s - rs^3 + 3rs + r^3 + r^4 + 2s^2)G_{n+3} + (4r^2 s + r^4 + 2s^2 + s^4)G_{n+2}$.

(b) $(-s)^2 H_n = (2s + r^2 + s^2)G_{n+3} - (2s + rs^2 + 3rs + r^2 + r^3)G_{n+2} + (3rs + r^3 - s^3)G_{n+1}$.

(c) $(-s)H_n = -(s-r)G_{n+2} + (rs - 2s - r - r^2)G_{n+1} + (r^2 + s^2 + 2s)G_n$.

(d) $H_n = 3G_{n+1} - 2(r+1)G_n - (s-r)G_{n-1}$.

(e) $H_n = (r+1)G_n + 2(s-r)G_{n-1} + 3(-s)G_{n-2}$.

(f) $s(r^2+4s)(r+s-1)^2 G_n = (-r^3+r^2 s+2r^2-3rs-r+4s^2+4s)H_{n+4} + (r^4-r^3 s-r^3+4r^2 s-r^2-4rs^2-3rs+r+2s^2-2s)H_{n+3} - (2s+4r^2 s+r^2 s^2-6rs+r^2-2r^3+r^4+2s^2+4s^3)H_{n+2}$.

(g) $-(r^2+4s)(r+s-1)^2 G_n = -2(r^2-r+3s+1)H_{n+3} + (-r-2s+7rs-r^2+2r^3+2)H_{n+2} + (-r+4s+r^2 s-3rs+2r^2-r^3+4s^2)H_{n+1}$.

(h) $-(r^2+4s)(r+s-1)^2 G_n = (rs-8s-r-r^2)H_{n+2} + (r+2s-r^2 s+5rs+r^3-2s^2)H_{n+1} + 2s(-r+3s+r^2+1)H_n$.

(i) $-(r^2+4s)(r+s-1)^2 G_n = -2(3s+rs+r^2+s^2)H_{n+1} + (2s+rs^2+5rs+r^2+r^3-2s^2)H_n + s(r+8s-rs+r^2)H_{n-1}$.

(j) $-(r^2+4s)(r+s-1)^2 G_n = -(4s+rs^2+2r^2 s+3rs+r^2+r^3+4s^2)H_n + (-rs^2+r^2 s+7rs+2r^3+2s^2-2s^3)H_{n-1} + 2s(3s+rs+r^2+s^2)H_{n-2}$.

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Lemma 36.]. □

Now, we give some basic relations between $\{G_n\}$ and $\{W_n\}$.

Lemma 5.2. *The following equalities are true:*

(a) $(-W_2^2-(r-s+1)W_1^2+sW_0^2+(r+2)W_1W_2-rW_0W_2+(r-2s)W_0W_1)(-W_2+rW_1+sW_0)G_n=((r+1)W_1^2+(-s)W_0^2-W_1W_2+(s-r)W_0W_1)W_{n+2}+(W_2^2-(r+1)W_1W_2-(s-r)W_0W_2-(-s)W_0W_1)W_{n+1}+((-s)W_1^2-(-s)W_0W_2)W_n.$

(b) $(-W_2^2-(r-s+1)W_1^2+sW_0^2+(r+2)W_1W_2-rW_0W_2+(r-2s)W_0W_1)(-W_2+rW_1+sW_0)G_n=(W_2^2+(r+1)^2W_1^2+(r+1)(-s)W_0^2-2(r+1)W_1W_2-(s-r)W_0W_2+((r+1)(s-r)-(-s))W_0W_1)W_{n+1}+(((-s)+(r+1)(s-r))W_1^2+(s-r)(-s)W_0^2-(s-r)W_1W_2-(-s)W_0W_2+(s-r)^2W_0W_1)W_n+((r+1)(-s)W_1^2+(-s)^2W_0^2-(-s)W_1W_2+(s-r)(-s)W_0W_1)W_{n-1}.$

(c) $(-W_2^2-(r-s+1)W_1^2+sW_0^2+(r+2)W_1W_2-rW_0W_2+(r-2s)W_0W_1)(-W_2+rW_1+sW_0)G_n=((r+1)W_2^2+((r+1)^3+(-s)+(r+1)(s-r))W_1^2+(-s)((s-r)+(r+1)^2)W_0^2-((s-r)+2(r+1)^2)W_1W_2-((-s)+(r+1)(s-r))W_0W_2+((r+1)^2(s-r)+(s-r)^2-(r+1)(-s))W_0W_1)W_n+((s-r)W_2^2+(r+1)((r+1)(s-r)+(-s))W_1^2+(-s)((-s)+(r+1)(s-r))W_0^2-((-s)+2(r+1)(s-r))W_1W_2-(s-r)^2W_0W_2+(r+1)(s-r)^2W_0W_1)W_{n-1}+((-s)W_2^2+(r+1)^2(-s)W_1^2+(r+1)(-s)^2W_0^2-2(r+1)(-s)W_1W_2-(s-r)(-s)W_0W_2+(-s)((r+1)(s-r)-(-s))W_0W_1)W_{n-2}.$

(d) $(-s)W_n=(W_2-(r+1)W_1-(s-r)W_0)G_{n+2}+(-(r+1)W_2+(r+1)^2W_1+((-s)+(r+1)(s-r))W_0)G_{n+1}+(-(s-r)W_2+((-s)+(r+1)(s-r))W_1+((s-r)^2-(r+1)(-s))W_0)G_n.$

(e) $W_n=W_0G_{n+1}+(W_1-(r+1)W_0)G_n+(W_2-(r+1)W_1-(s-r)W_0)G_{n-1}.$

(f) $W_n=W_1G_n+(W_2-(r+1)W_1)G_{n-1}+(-s)W_0G_{n-2}.$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Lemma 37.]. □

Next, we present some basic relations between $\{H_n\}$ and $\{W_n\}$.

Lemma 5.3. *The following equalities hold:*

(a) $(-W_2^2-(r-s+1)W_1^2+sW_0^2+(r+2)W_1W_2-rW_0W_2+(r-2s)W_0W_1)(-W_2+rW_1+sW_0)H_n=(3W_2^2+((r+1)^2-(s-r))W_1^2+(r+1)(-s)W_0^2-4(r+1)W_1W_2-2(s-r)W_0W_2+((r+1)(s-r)-3(-s))W_0W_1)W_{n+2}+(-2(r+1)W_2^2+3(-s)W_1^2-2(s-r)W_1W_2-3(-s)W_0W_2+3(r+1)(s-r)W_1^2+2(s-r)(-s)W_0^2+2(r+1)^2W_1W_2+2(s-r)^2W_0W_1+(r+1)(s-r)W_0W_2+2(r+1)(-s)W_0W_1)W_{n+1}+(-(s-r)W_2^2+((s-r)^2+(r+1)(-s))W_1^2+3(-s)^2W_0^2+((r+1)(s-r)-3(-s))W_1W_2+2(r+1)(-s)W_0W_2+4(s-r)(-s)W_0W_1)W_n.$

(b) $(-W_2^2-(r-s+1)W_1^2+sW_0^2+(r+2)W_1W_2-rW_0W_2+(r-2s)W_0W_1)(-W_2+rW_1+sW_0)H_n=((r+1)^3W_1^2+(r+1)W_2^2+3(-s)W_1^2+(r+1)^2(-s)W_0^2-2(s-r)W_1W_2-3(-s)W_0W_2+2(r+1)(s-r)W_1^2+2(s-r)(-s)W_0^2-2(r+1)^2W_1W_2+2(s-r)^2W_0W_1-(r+1)(s-r)W_0W_2-(r+1)(-s)W_0W_1+(r+1)^2(s-r)W_0W_1)W_{n+1}+(3(-s)^2W_0^2+2(s-r)W_2^2+(r+1)^2(s-r)W_1^2-3(-s)W_1W_2+(r+1)(-s)W_1^2-2(s-r)^2W_0W_2-3(r+1)(s-r)W_1W_2+2(r+1)(-s)W_0W_2+(s-r)(-s)W_0W_1+(r+1)(s-r)(-s)W_0^2+(r+1)(s-r)^2W_0W_1)W_n+(3(-s)W_2^2+(-s)((r+1)^2-(s-r))W_1^2+(r+1)(-s)^2W_0^2-4(r+1)(-s)W_1W_2-2(s-r)(-s)W_0W_2+(-s)((r+1)(s-r)-3(-s))W_0W_1)W_{n-1}.$

(c) $(-W_2^2-(r-s+1)W_1^2+sW_0^2+(r+2)W_1W_2-rW_0W_2+(r-2s)W_0W_1)(-W_2+rW_1+sW_0)H_n=(((r+1)^2+2(s-r))W_2^2+(r+1)((r+1)^3+3(r+1)(s-r)+4(-s))W_1^2+(-s)((r+1)^3+3(r+1)(s-r)+3(-s))W_0^2-(3(-s)+2(r+1)^3+5(r+1)(s-r))W_1W_2-((r+1)^2(s-r)+(r+1)(-s)+2(s-r)^2)W_0W_2+((r+1)^3(s-r)-(r+1)^2(-s)+3(r+1)(s-r)^2+(s-r)(-s))W_0W_1)W_n+((3(-s)+(r+1)(s-r))W_2^2+((r+1)^3(s-r)+2(r+1)(s-r)^2+2(s-r)(-s)+(r+1)^2(-s))W_1^2+(-s)((r+1)(-s)+(r+1)^2(s-r)+2(s-r)^2)W_0^2-2(2(r+1)(-s)+(s-r)^2+(r+1)^2(s-r))W_1W_2-(s-r)((r+1)(s-r)+5(-s))W_0W_2+(2(s-r)^3+(r+1)^2(s-r)^2-3(-s)^2)W_0W_1)W_{n-1}+((r+1)(-s)W_2^2+(-s)((r+1)^3+3(-s)+2(r+1)(s-r))W_1^2+(-s)^2(r^2+2s+1)W_0^2-2(-s)((s-r)+(r+1)^2)W_1W_2-(-s)(3(-s)+(r+1)(s-r))W_0W_2+(-s)(2(s-r)^2+(r+1)^2(s-r)-(r+1)(-s))W_0W_1)W_{n-2}.$

(d) $-(r^2+4s)(r+s-1)^2W_n = (-2((r+1)^2+3(s-r))W_2+(2(r+1)^3+9(-s)+7(r+1)(s-r))W_1+(4(s-r)^2-3(r+1)(-s)+(r+1)^2(s-r))W_0)H_{n+2}+((2(r+1)^3+9(-s)+7(r+1)(s-r))W_2-2((r+1)^4+4(r+1)^2(s-r)+6(-s)(r+1)+(s-r)^2)W_1-(4(r+1)(s-r)^2+6(-s)(s-r)-(-s)(r+1)^2+(r+1)^3(s-r))W_0)H_{n+1}+((-3(r+1)(-s)+4(s-r)^2+(r+1)^2(s-r))W_2-(4(r+1)(s-r)^2+(r+1)^3(s-r)-(r+1)^2(-s)+6(s-r)(-s))W_1+(-(r+1)^2(s-r)^2+2(r+1)^3(-s)+9(-s)^2-4(s-r)^3+10(r+1)(s-r)(-s))W_0)H_n.$

(e) $-(r^2+4s)(r+s-1)^2W_n = ((9(-s)+(r+1)(s-r))W_2-((r+1)^2(s-r)+3(-s)(r+1)+2(s-r)^2)W_1-2(-s)(3(s-r)+(r+1)^2)W_0)H_{n+1}+(-((r+1)^2(s-r)+2(s-r)^2+3(r+1)(-s))W_2+((r+1)^3(s-r)+3(r+1)(s-r)^2+(r+1)^2(-s)+3(s-r)(-s))W_1+(-s)(9(-s)+2(r+1)^3+7(r+1)(s-r))W_0)H_n+(-2(-s)((r+1)^2+3(s-r))W_2+(-s)(2(r+1)^3+9(-s)+7(r+1)(s-r))W_1+(-s)(4(s-r)^2+(r+1)^2(s-r)-3(r+1)(-s))W_0)H_{n-1}.$

(f) $-(r^2+4s)(r+s-1)^2W_n = (2(3(r+1)(-s)-(s-r)^2)W_2+(3(s-r)(-s)+(r+1)(s-r)^2-2(r+1)^2(-s))W_1+(-s)(9(-s)+(r+1)(s-r))W_0)H_n+(((r+1)(s-r)^2-2(r+1)^2(-s)+3(s-r)(-s))W_2+(2(r+1)^3(-s)-(r+1)^2(s-r)^2+4(r+1)(s-r)(-s)-2(s-r)^3+9(-s)^2)W_1-(-s)(2(s-r)^2+3(r+1)(-s)+(r+1)^2(s-r))W_0)H_{n-1}+((-s)(9(-s)+(r+1)(s-r))W_2-(-s)((r+1)^2(s-r)+3(r+1)(-s)+2(s-r)^2)W_1-2(-s)^2(3(s-r)+(r+1)^2)W_0)H_{n-2}.$

Proof. Replace r, s and t with $r+1,\ s-r,\ -s$, respectively, in [25, Lemma 38.]. □

6 RECURRENCE PROPERTIES OF GENERALIZED HORADAM-LEONARDO POLYNOMIALS

Now, we give a formula for W_{-n}.

Theorem 6.1. *For $n \in \mathbb{Z}$, we have*

$$W_{-n} = (-s)^{-n}(W_{2n} - H_nW_n + \frac{1}{2}(H_n^2 - H_{2n})W_0).$$

Proof. Replace r, s and t with $r+1,\ s-r,\ -s$, respectively, in [25, Theorem 39.]. □

Next, we present the following corollary.

Corollary 6.2. *For $n \in \mathbb{Z}$, we get*

(a) $G_{-n} = \frac{1}{(-s)^{n+1}}((2(r+1)(-s)-(s-r)^2)G_n^2+(-s)G_{2n}+(s-r)G_{n+2}G_n-(3(-s)+(r+1)(s-r))G_{n+1}G_n).$

(b) $H_{-n} = \frac{1}{2(-s)^n}(H_n^2 - H_{2n}).$

Proof. Replace r, s and t with $r+1, s-r, -s$, respectively, in [25, Corollary 42] or put $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = (r+1)$ and take $W_n = H_n$ with $H_0 = 3, H_1 = (r+1), H_2 = r^2+2s+1$, respectively, in the last Theorem. □

7 GENERALIZED HORADAM-LEONARDO POLYNOMIALS BY MATRIX METHODS

In this section, we give matrix representations of the sequences $W_n, G_n,\ H_n$ and present Simson matrix and investigate its properties.

7.1 Matrix related to the Sequences W_n, G_n and H_n

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} r+1 & s-r & -s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = (-s)$. We also define

$$B_n = \begin{pmatrix} G_{n+1} & (s-r)G_n + (-s)G_{n-1} & (-s)G_n \\ G_n & (s-r)G_{n-1} + (-s)G_{n-2} & (-s)G_{n-1} \\ G_{n-1} & (s-r)G_{n-2} + (-s)G_{n-3} & (-s)G_{n-2} \end{pmatrix}$$

and

$$D_n = \begin{pmatrix} W_{n+1} & (s-r)W_n + (-s)W_{n-1} & (-s)W_n \\ W_n & (s-r)W_{n-1} + (-s)W_{n-2} & (-s)W_{n-1} \\ W_{n-1} & (s-r)W_{n-2} + (-s)W_{n-3} & (-s)W_{n-2} \end{pmatrix}.$$

Theorem 7.1. *The following properties hold: for all integers m, n,*

(a) $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} r+1 & s-r & -s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & (s-r)G_n + (-s)G_{n-1} & (-s)G_n \\ G_n & (s-r)G_{n-1} + (-s)G_{n-2} & (-s)G_{n-1} \\ G_{n-1} & (s-r)G_{n-2} + (-s)G_{n-3} & (-s)G_{n-2} \end{pmatrix}.$$

(b) $D_1 A^n = A^n D_1$.

(c) $D_{n+m} = D_n B_m = B_m D_n$, *i.e.,*

(d)

$$A^n = G_{n-1}A^2 + ((s-r)G_{n-2} + (-s)G_{n-3})A + (-s)G_{n-2}I,$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Theorem 51.]. □

Now, we give matrix formulas for the generalized Horadam-Leonardo polynomials and (r, s)-Horadam-Leonardo-Lucas polynomials.

Corollary 7.2. *For all integers n, the following formulas for generalized Horadam-Leonardo polynomials and (r, s)-Horadam-Leonardo-Lucas polynomials hold.*

(a) *(Generalized Horadam-Leonardo polynomials).*

$$\begin{pmatrix} r+1 & s-r & -s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{\Lambda_W(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$a_{11} = ((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+3} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n+2} + ((-s)W_1^2 - (-s)W_0W_2)W_{n+1}$,

$a_{21} = ((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+2} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n+1} + ((-s)W_1^2 - (-s)W_0W_2)W_n$,

$a_{31} = ((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+1} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_n + ((-s)W_1^2 - (-s)W_0W_2)W_{n-1},$

$a_{12} = (s-r)(((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+2} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n+1} + ((-s)W_1^2 - (-s)W_0W_2)W_n) + (-s)(((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+1} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_n + ((-s)W_1^2 - (-s)W_0W_2)W_{n-1}),$

$a_{22} = (s-r)(((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+1} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_n + ((-s)W_1^2 - (-s)W_0W_2)W_{n-1}) + (-s)(((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_n + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n-1} + ((-s)W_1^2 - (-s)W_0W_2)W_{n-2}),$

$a_{32} = (s-r)(((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_n + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n-1} + ((-s)W_1^2 - (-s)W_0W_2)W_{n-2}) + (-s)(((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n-1} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n-2} + ((-s)W_1^2 - (-s)W_0W_2)W_{n-3}),$

$a_{13} = (-s)(((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+2} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n+1} + ((-s)W_1^2 - (-s)W_0W_2)W_n),$

$a_{23} = (-s)(((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+1} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_n + ((-s)W_1^2 - (-s)W_0W_2)W_{n-1}),$

$a_{33} = (-s)(((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_n + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n-1} + ((-s)W_1^2 - (-s)W_0W_2)W_{n-2}),$

$\Lambda_W(0)$ *given as in (4.2), i.e.,*

$$\begin{aligned}\Lambda_W(0) &= W_2^3 - r(r-s+1)W_1^3 + (-s)^2W_0^3 - 2(r+1)W_1W_2^2 - (s-r)W_0W_2^2 \\ &\quad +(r^2+3r-s+1)W_2W_1^2 + ((r+1)(-s) + (s-r)^2)W_0W_1^2 \\ &\quad +(r+1)(-s)W_0^2W_2 + 2(s-r)(-s)W_0^2W_1 - (r^2+r-4s-rs)W_2W_1W_0.\end{aligned}$$

(b) *((r,s)-Horadam-Leonardo-Lucas polynomials).*

$$\begin{pmatrix} r+1 & s-r & -s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n$$

$$= \frac{1}{4(r+1)^3(-s) - (r+1)^2(s-r)^2 - 4(s-r)^3 + 27(-s)^2 + 18(r+1)(s-r)(-s)} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

where

$b_{11} = (9(-s) + (r+1)(s-r))H_{n+3} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n+2} - 2(-s)((r+1)^2 + 3(s-r))H_{n+1},$

$b_{21} = (9(-s) + (r+1)(s-r))H_{n+2} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n+1} - 2(-s)((r+1)^2 + 3(s-r))H_n,$

$b_{31} = (9(-s) + (r+1)(s-r))H_{n+1} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_n - 2(-s)((r+1)^2 + 3(s-r))H_{n-1},$

$b_{12} = (s-r)((9(-s) + (r+1)(s-r))H_{n+2} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n+1} - 2(-s)((r+1)^2 + 3(s-r))H_n) + (-s)((9(-s) + (r+1)(s-r))H_{n+1} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_n - 2(-s)((r+1)^2 + 3(s-r))H_{n-1}),$

$b_{22} = (s-r)((9(-s) + (r+1)(s-r))H_{n+1} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_n - 2(-s)((r+1)^2 + 3(s-r))H_{n-1}) + (-s)((9(-s) + (r+1)(s-r))H_n - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n-1} - 2(-s)((r+1)^2 + 3(s-r))H_{n-2}),$

$b_{32} = (s-r)((9(-s) + (r+1)(s-r))H_n - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n-1} - 2(-s)((r+1)^2 + 3(s-r))H_{n-2}) + (-s)((9(-s) + (r+1)(s-r))H_{n-1} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n-2} - 2(-s)((r+1)^2 + 3(s-r))H_{n-3}),$

$b_{13} = (-s)((9(-s) + (r+1)(s-r))H_{n+2} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n+1} - 2(-s)((r+1)^2 + 3(s-r))H_n),$

$b_{23} = (-s)((9(-s) + (r+1)(s-r))H_{n+1} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_n - 2(-s)((r+1)^2 + 3(s-r))H_{n-1})$,
$b_{33} = (-s)((9(-s) + (r+1)(s-r))H_n - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n-1} - 2(-s)((r+1)^2 + 3(s-r))H_{n-2})$.

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Corollary 52]. □

Next, we give an identity for W_{n+m}.

Theorem 7.3. *(Honsberger Formula) We have*

$$\begin{aligned} W_{n+m} &= W_n G_{m+1} + W_{n-1}((s-r)G_m + (-s)G_{m-1}) + (-s)W_{n-2}G_m \\ &= W_n G_{m+1} + ((s-r)W_{n-1} + (-s)W_{n-2})\, G_m + (-s)W_{n-1}G_{m-1}, \end{aligned}$$

for all integers m and n.

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Theorem 53]. □

Corollary 7.4. *We have the following properties:*

$$\begin{aligned} G_{n+m} &= G_n G_{m+1} + G_{n-1}((s-r)G_m + (-s)G_{m-1}) + (-s)G_{n-2}G_m, \\ H_{n+m} &= H_n G_{m+1} + H_{n-1}((s-r)G_m + (-s)G_{m-1}) + (-s)H_{n-2}G_m, \end{aligned}$$

for all integers m and n.

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Corollary 54]. □

Corollary 7.5. *We have the following properties:*

$$\begin{aligned} W_{mn+j} &= G_{mn-1}W_{j+2} + ((s-r)G_{mn-2} + (-s)G_{mn-3})\, W_{j+1} + (-s)G_{mn-2}W_j, \\ G_{mn+j} &= G_{mn-1}G_{j+2} + ((s-r)G_{mn-2} + (-s)G_{mn-3})\, G_{j+1} + (-s)G_{mn-2}G_j, \\ H_{mn+j} &= G_{mn-1}H_{j+2} + ((s-r)G_{mn-2} + (-s)G_{mn-3})\, H_{j+1} + (-s)G_{mn-2}H_j, \end{aligned}$$

for all integers m and n.

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Corollary 55.]. □

7.2 The Simson Matrix

For $n \in \mathbb{Z}$, we define

$$f_W(n) = \begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix}.$$

Let us call this matrix as Simson matrix of the sequence W_n. Then, as special cases of W_n,

$$f_G(n) = \begin{pmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{pmatrix},\ f_H(n) = \begin{pmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{pmatrix},$$

are Simson matrices of the sequences G_n and H_n,respectively.

Lemma 7.6. *The followings hold: For all integers n, m and j,*

(a) $f_W(n) = (r+1)f_W(n-1) + (s-r)f_W(n-2) + (-s)f_W(n-3)$.

(b)

$$\begin{aligned} f_W(n) &= Af_W(n-1), \\ f_W(n) &= A^n f_W(0). \end{aligned}$$

(c)

$$\begin{aligned} f_W(n+m) &= A^n f_W(m), \\ f_W(n+m) &= A^m f_W(n), \\ f_W(n) &= A^m f_W(n-m). \end{aligned}$$

(d)

$$f_W(mn+j) = A^{mn} f_W(j)$$

and

$$f_W(mn+j) = (G_{n-1}A^2 + ((s-r)G_{n-2} + (-s)G_{n-3})A + (-s)G_{n-2}I)^m f_W(j).$$

(e)

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \begin{pmatrix} G_{n+1} & (s-r)G_n + (-s)G_{n-1} & (-s)G_n \\ G_n & (s-r)G_{n-1} + (-s)G_{n-2} & (-s)G_{n-1} \\ G_{n-1} & (s-r)G_{n-2} + (-s)G_{n-3} & (-s)G_{n-2} \end{pmatrix} \begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{pmatrix}.$$

(f)

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \frac{1}{\Lambda_W(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{pmatrix}$$

where $a_{33}, a_{23}, a_{13}, a_{32}, a_{22}, a_{12}, a_{31}, a_{21}, a_{11},$ *and* $\Lambda_W(0)$ *are as in Corollary 7.2 (a) (in the last identity above, we replace* n *with* m *in* $a_{33}, a_{23}, a_{13}, a_{32}, a_{22}, a_{12}, a_{31}, a_{21}, a_{11}$*).*

(g)

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \frac{\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{pmatrix}}{-(r^2+4s)(r+s-1)^2}$$

where $b_{33}, b_{23}, b_{13}, b_{32}, b_{22}, b_{12}, b_{31}, b_{21}, b_{11}$ *are as in Corollary 7.2 (b) (in the last identity above, we replace* n *with* m *in* $b_{33}, b_{23}, b_{13}, b_{32}, b_{22}, b_{12}, b_{31}, b_{21}, b_{11}$*).*

Proof. Replace r, s and t with $r+1,\ s-r,\ -s,$ respectively, in [25, Lemma 56]. □

By taking the determinant of both sides of the identities given in Lemma 7.6, one get the following Theorem.

Theorem 7.7. *For all* $n, m \in \mathbb{Z}$*, the following identities are true.*

(a) *Catalan's Formula:*

$$\det(f_W(n+m)) = (-s)^n \det(f_W(m)),\ \ \det(f_W(n)) = (-s)^m \det(f_W(n-m)).$$

(b) *Cassini's(or Simson's) Formula (see Theorem 4.1):*

$$\det(f_W(n)) = (-s)^n \det(f_W(0)).$$

Proof. Replace r, s and t with $r+1,\ s-r,\ -s,$ respectively, in [25, Theorem 57]. □

By using the last Theorem, one get the following Corollary which produces determinantal formulas of (r, s)-Horadam-Leonardo polynomials (put $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = (r+1)$).

Corollary 7.8. *For all $n, m \in \mathbb{Z}$, the following identities are true.*

(a) *Catalan's Formula:*

$$\det(f_G(n+m)) = (-s)^n \det(f_G(m)), \det(f_G(n)) = (-s)^m \det(f_G(n-m)).$$

(b) *Cassini's(or Simson's) Formula:*

$$\det(f_G(n)) = (-s)^n \det(f_G(0)),\ i.e., \begin{vmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{vmatrix} = -(-s)^{n-1}.$$

By putting $W_n = H_n$ with $H_0 = 3, H_1 = (r+1), H_2 = r^2 + 2s + 1$ in the last Theorem, one get the following Corollary which produces determinantal formulas of (r,s)-Horadam-Leonardo-Lucas polynomials.

Corollary 7.9. *For all $n, m \in \mathbb{Z}$, the following identities are true.*

(a) *Catalan's Formula:*

$$\det(f_H(n+m)) = (-s)^n \det(f_H(m)), \det(f_H(n)) = (-s)^m \det(f_H(n-m)).$$

(b) *Cassini's(or Simson's) Formula:*

$$\det(f_H(n)) = (-s)^n \det(f_H(0)).$$

8 THE SUM FORMULA $\sum_{k=0}^{n} z^k W_{mk+j}$ OF GENERALIZED HORADAM-LEONARDO POLYNOMIALS

Now, we present the sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Horadam-Leonardo polynomials.

8.1 The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of Generalized Horadam-Leonardo Polynomials in Terms of Generalized Horadam-Leonardo Polynomials

We can give the sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Horadam-Leonardo polynomials (in terms of elements of the sequence of generalized Horadam-Leonardo polynomials).

Theorem 8.1. *For all $m, j \in \mathbb{Z}$, one get the following sum formulas.*

(a) *If $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 \neq 0$ then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4} = \frac{\Theta_W(z)}{\Gamma_W(z)}$$

where

$\Theta_W(z) = z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6,$

$z^{n+3}\Theta_1 = z^{n+3}(-W_jW_{m+2}^2W_{m+mn+2} + (-W_{j+1} + (r+1)W_j)W_{m+2}^2W_{m+mn+1} + (-W_{j+2} + (r+1)W_{j+1} + (s-r)W_j)W_{m+2}^2W_{m+mn} + (-W_{j+2} + (s-r)W_j)W_{m+1}^2W_{m+mn+2} - ((-s) + (r+1)(s-r))W_jW_{m+1}^2W_{m+mn+1} + ((s-r)W_{j+2} - ((-s) + (r+1)(s-r))W_{j+1} - (s-r)^2W_j)W_{m+1}^2W_{m+mn} - (-s)W_m^2W_{j+1}W_{m+mn+2} + (-s)(-W_{j+2} + (r+1)W_{j+1})W_m^2W_{m+mn+1} - (-s)^2W_jW_m^2W_{m+mn} + (W_{j+1} + (r+1)W_j)W_{m+2}W_{m+1}W_{m+mn+2} + (W_{j+2} - (r+1)W_{j+1})W_{m+2}W_mW_{m+mn+2} + (-(s-r)W_{j+1} + (-s)W_j)W_{m+1}W_mW_{m+mn+2} + (W_{j+2} - (r+1)^2W_j)W_{m+2}W_{m+1}W_{m+mn+1} + (-(r+1)W_{j+2} + ((r+1)^2 + (s-r))W_{j+1} + (-s)W_j)W_{m+2}W_m W_{m+mn+1} + (-(s-r)W_{j+2} + ((-s) + (r+1)(s-r))W_{j+1} - (r+1)(-s)W_j)W_{m+1}W_mW_{m+mn+1} + (r+1)W_{j+2}W_{m+1}W_{m+2}W_{m+mn} + (-s)W_jW_{m+1}W_{m+2}W_{m+mn} - (r+1)(s-r)W_jW_{m+1}W_{m+2}W_{m+mn} - (r+$

$1)^2W_{j+1}W_{m+1}W_{m+2}W_{m+mn}+(-s)(W_{j+1}-(r+1)W_j)W_{m+2}W_mW_{m+mn}+(-s)(W_{j+2}-(r+1)W_{j+1}-2(s-r)W_j)W_{m+1}W_mW_{m+mn}),$

$z^{n+2}\Theta_2=z^{n+2}((-W_0W_{j+2}+((r+1)W_0-W_1)W_{j+1}+(2W_2-(r+1)W_1)W_j)W_{m+2}W_{m+mn+2}+(2W_1W_{j+2}+((s-r)W_0-W_2)W_{j+1}-((r+1)W_2+2(s-r)W_1+(-s)W_0)W_j)W_{m+1}W_{m+mn+2}+(-W_2W_{j+2}+((r+1)W_2+(s-r)W_1+2(-s)W_0)W_{j+1}-(-s)W_1W_j)W_mW_{m+mn+2}+(((r+1)W_0-W_1)W_{j+2}+(2W_2-(r+1)^2W_0-(s-r)W_0)W_{j+1}+(-2(r+1)W_2+(r+1)^2W_1-(-s)W_0)W_j)W_{m+2}W_{m+mn+1}+((-W_2+(s-r)W_0)W_{j+2}-((r+1)(s-r)W_0+(-s)W_0)W_{j+1}+((r+1)^2W_2+2((-s)+(r+1)(s-r))W_1+(r+1)(-s)W_0)W_j)W_{m+1}W_{m+mn+1}+(((r+1)W_2+(s-r)W_1+2(-s)W_0)W_{j+2}-((r+1)^2W_2+(r+1)(s-r)W_1+2(r+1)(-s)W_0+(s-r)W_2+(-s)W_1)W_{j+1}+(-s)((r+1)W_1-W_2)W_j)W_mW_{m+mn+1}+((2W_2-(r+1)W_1)W_{j+2}+((r+1)^2W_1-2(r+1)W_2-(-s)W_0)W_{j+1}+((r+1)(s-r)W_1-2(s-r)W_2-(-s)W_1+(r+1)(-s)W_0)W_j)W_{m+2}W_{m+mn}+(-((r+1)W_2+2(s-r)W_1+(-s)W_0)W_{j+2}+((r+1)^2W_2+2((-s)+(r+1)(s-r))W_1+(r+1)(-s)W_0)W_{j+1}+(((r+1)(s-r)-(-s))W_2+2(s-r)^2W_1+2(s-r)(-s)W_0)W_j)W_{m+1}W_{m+mn}+(-s)(-W_1W_{j+2}+((r+1)W_1-W_2)W_{j+1}+((r+1)W_2+2(s-r)W_1+2(-s)W_0)W_j)W_mW_{m+mn}),$

$z^{n+1}\Theta_3=z^{n+1}(((W_0W_2-W_1^2)W_{j+2}+(-(-s)W_0^2+W_1W_2-(r+1)W_0W_2-(s-r)W_0W_1)W_{j+1}+(-W_2^2+(s-r)W_1^2+(r+1)W_1W_2+(-s)W_0W_1)W_j)W_{m+mn+2}+((-(-s)W_0^2+W_1W_2-(r+1)W_0W_2-(s-r)W_0W_1)W_{j+2}+(-W_2^2+(r+1)(-s)W_0^2+((r+1)^2+(s-r))W_0W_2+((-s)+(r+1)(s-r))W_0W_1)W_{j+1}+((r+1)W_2^2-((r+1)(s-r)+(-s))W_1^2-(r+1)^2W_1W_2+(-s)W_0W_2-(r+1)(-s)W_0W_1)W_j)W_{m+mn+1}+((-W_2^2+(s-r)W_1^2+(r+1)W_1W_2+(-s)W_0W_1)W_{j+2}+((r+1)W_2^2-((r+1)(s-r)+(-s))W_1^2-(r+1)^2W_1W_2+(-s)W_0W_2-(r+1)(-s)W_0W_1)W_{j+1}+((s-r)W_2^2-(s-r)^2W_1^2-(-s)^2W_0^2+((-s)-(r+1)(s-r))W_1W_2-(r+1)(-s)W_0W_2-2(s-r)(-s)W_0W_1)W_j)W_{m+mn}),$

$z^2\Theta_4=z^2((W_0W_{j+2}+(W_1-(r+1)W_0)W_{j+1}+(W_2-(r+1)W_1-(s-r)W_0)W_j)W_{m+2}^2+((W_2-(s-r)W_0)W_{j+2}+((-s)W_0+(r+1)(s-r)W_0)W_{j+1}+((s-r)^2W_0+((r+1)(s-r)+(-s))W_1-(s-r)W_2)W_j)W_{m+1}^2+(-s)(W_1W_{j+2}+(W_2-(r+1)W_1)W_{j+1}+(-s)W_0W_j)W_m^2+(-(W_1+(r+1)W_0)W_{j+2}+((r+1)^2W_0-W_2)W_{j+1}+(-(r+1)W_2+(r+1)^2W_1+((r+1)(s-r)-(-s))W_0)W_j)W_{m+1}W_{m+2}+(((r+1)W_1-W_2)W_{j+2}+((r+1)W_2-((s-r)+(r+1)^2)W_1-(-s)W_0)W_{j+1}+(-s)((r+1)W_0-W_1)W_j)W_{m+2}W_m+(((s-r)W_1-(-s)W_0)W_{j+2}+((s-r)W_2-((r+1)(s-r)+(-s))W_1+(r+1)(-s)W_0)W_{j+1}+(-s)(-W_2+(r+1)W_1+2(s-r)W_0)W_j)W_{m+1}W_m),$

$z\Theta_5=z(((W_1^2-W_0W_2)W_{j+2}+((-s)W_0^2-W_1W_2+(r+1)W_0W_2+(s-r)W_0W_1)W_{j+1}+(-2W_2^2-(r+1)^2W_1^2-(-s)(r+1)W_0^2+3(r+1)W_1W_2+2(s-r)W_0W_2+(2(-s)-(s-r)(r+1))W_0W_1)W_j)W_{m+2}+(((-s)W_0^2-W_1W_2+(r+1)W_0W_2+(s-r)W_0W_1)W_{j+2}+(W_2^2-((r+1)^2+(s-r))W_0W_2-((r+1)(s-r)+(-s))W_0W_1)W_{j+1}+((r+1)W_2^2-2((-s)+(r+1)(s-r))W_1^2-2(s-r)(-s)W_0^2+(2(s-r)-(r+1)^2)W_1W_2+(2(-s)-(r+1)(s-r))W_0W_2-(2(s-r)^2+(r+1)(-s))W_0W_1)W_j)W_{m+1}+((W_2^2-(s-r)W_1^2-(r+1)W_1W_2-(-s)W_0W_1)W_{j+2}+(-(r+1)W_2^2+((r+1)(s-r)+(-s))W_1^2+(r+1)^2W_1W_2-(-s)W_0W_2+(r+1)(-s)W_0W_1)W_{j+1}+(-s)(-(r+1)W_1^2-2(-s)W_0^2+2W_1W_2-(r+1)W_0W_2-2(s-r)W_0W_1)W_j)W_m-(r+1)(-s)W_0^2W_{j+1}W_{m+1}),$

$\Theta_6=(W_2^3+((-s)+(r+1)(s-r))W_1^3+(-s)^2W_0^3-2(r+1)W_1W_2^2-(s-r)W_0W_2^2+((r+1)^2-(s-r))W_1^2W_2+((s-r)^2+(r+1)(-s))W_0W_1^2+(r+1)(-s)W_0^2W_2+2(s-r)(-s)W_0^2W_1+((r+1)(s-r)-3(-s))W_0W_1W_2)W_j,$

and

$\Gamma_W(z)=z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4,$

$z^3\Gamma_1=z^3(-(-s)^m(W_2^3+((-s)+(r+1)(s-r))W_1^3+(-s)^2W_0^3+((r+1)^2-(s-r))W_1^2W_2-2(r+1)W_1W_2^2-(s-r)W_0W_2^2+(r+1)(-s)W_0^2W_2+((s-r)^2+(r+1)(-s))W_0W_1^2+2(s-r)(-s)W_0^2W_1+((r+1)(s-r)-3(-s))W_0W_1W_2)),$

$z^2\Gamma_2=z^2((3W_2-2(r+1)W_1-(s-r)W_0)W_{m+2}^2+(((r+1)^2-(s-r))W_2+(3(r+1)(s-r)+3(-s))W_1+((s-r)^2+(r+1)(-s))W_0)W_{m+1}^2+(-s)((r+1)W_2+2(s-r)W_1+3(-s)W_0)W_m^2+(-4(r+1)W_2+2((r+1)^2-(s-r))W_1+((r+1)(s-r)-3(-s))W_0)W_{m+2}W_{m+1}+(-2(s-r)W_2+((r+1)(s-r)-3(-s))W_1+2(r+1)(-s)W_0)W_{m+2}W_m+(((r+1)(s-r)-3(-s))W_2+2((s-r)^2+(r+1)(-s))W_1+4(s-r)(-s)W_0)W_{m+1}W_m),$

$z\Gamma_3=z((-3W_2^2+((s-r)-(r+1)^2)W_1^2-(-s)(r+1)W_0^2+4(r+1)W_1W_2+2(s-r)W_0W_2+(3(-s)-(s-r)(r+1))W_0W_1)W_{m+2}+(2(r+1)W_2^2-(3(r+1)(s-r)+3(-s))W_1^2-2(s-r)(-s)W_0^2+(2(s-r)-2(r+1)^2)W_1W_2+(3(-s)-(r+1)(s-r))W_0W_2-2((s-r)^2+(r+1)(-s))W_0W_1)W_{m+1}+((s-r)W_2^2-((s-r)^2+$

$(r+1)(-s))W_1^2 - 3(-s)^2W_0^2 + (3(-s) - (r+1)(s-r))W_1W_2 - 2(r+1)(-s)W_0W_2 - 4(s-r)(-s)W_0W_1)$ $W_m)$,

$\Gamma_4 = W_2^3 + ((-s)+(r+1)(s-r))W_1^3 + (-s)^2W_0^3 - 2(r+1)W_1W_2^2 + ((r+1)^2 - (s-r))W_1^2W_2 - (s-r)W_0W_2^2 + ((s-r)^2 + (r+1)(-s))W_0W_1^2 + (r+1)(-s)W_0^2W_2 + 2(s-r)(-s)W_0^2W_1 + ((r+1)(s-r) - 3(-s))W_0W_1W_2.$

(b) *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)(z-b)(z-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(c) *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^2(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z = a$ *or* $z = b$ *then for* $z = a$ *we get*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)(n+2)z^{n+1}\Theta_1 + (n+2)(n+1)z^n\Theta_2 + (n+1)nz^{n-1}\Theta_3 + 2\Theta_4}{6z\Gamma_1 + 2\Gamma_2},$$

and for $z = b$ *we get*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(d) *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)(n+2)(n+1)z^n\Theta_1 + (n+2)(n+1)nz^{n-1}\Theta_2 + (n+1)n(n-1)z^{n-2}\Theta_3}{6\Gamma_1}.$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Theorem 61]. □

Next, we present some special cases of the last Theorem.

Theorem 8.2. *One get the following summing formulas.*

(a) $(m = 1, j = 0)$.

(i) *If* $z^3(-(-s)) + z^2(-1)(s-r) + z(-1)(r+1) + 1 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_1}{z^3(-(-s)) + z^2(-1)(s-r) + z(-1)(r+1) + 1}$$

where

$\Omega_1 = z^{n+3}(-1)(-s)W_n + z^{n+2}((r+1)W_{n+1} - W_{n+2}) + z^{n+1}(-1)W_{n+1} + z^2(W_2 - (r+1)W_1 - (s-r)W_0) + z(W_1 - (r+1)W_0) + W_0.$

(ii) *If* $z^3(-(-s)) + z^2(-1)(s-r) + z(-1)(r+1) + 1 = u(z-a)(z-b)(z-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_2}{3z^2(-(-s)) + 2z(-1)(s-r) + (-1)(r+1)}$$

where

$\Omega_2 = (n+3)z^{n+2}(-1)(-s)W_n + (n+2)z^{n+1}((r+1)W_{n+1} - W_{n+2}) + (n+1)z^n(-1)W_{n+1} + 2z(W_2 - (r+1)W_1 - (s-r)W_0) + (W_1 - (r+1)W_0).$

(iii) *If* $z^3(-(-s))+z^2(-1)(s-r)+z(-1)(r+1)+1=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$*, i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_3}{6z(-(-s))+2(-1)(s-r)}$$

where

$\Omega_3=(n+3)(n+2)z^{n+1}(-1)(-s)W_n+(n+2)(n+1)z^n((r+1)W_{n+1}-W_{n+2})+(n+1)nz^{n-1}(-1)W_{n+1}+2(W_2-(r+1)W_1-(s-r)W_0)$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_4}{3z^2(-(-s))+2z(-1)(s-r)+(-1)(r+1)}$$

where

$\Omega_4=(n+3)z^{n+2}(-1)(-s)W_n+(n+2)z^{n+1}((r+1)W_{n+1}-W_{n+2})+(n+1)z^n(-1)W_{n+1}+2z(W_2-(r+1)W_1-(s-r)W_0)+(W_1-(r+1)W_0)$.

(iv) *If* $z^3(-(-s))+z^2(-1)(s-r)+z(-1)(r+1)+1=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$*, i.e.,* $z=a$*, then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_5}{6(-(-s))}$$

where

$\Omega_5=(n+3)(n+2)(n+1)z^n(-1)(-s)W_n+(n+2)(n+1)nz^{n-1}((r+1)W_{n+1}-W_{n+2})+(n+1)n(n-1)z^{n-2}(-1)W_{n+1}$.

(b) $(m=2,j=0)$.

(i) *If* $z^3(-(-s)^2)+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)(r^2+2s+1)+1\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_1}{z^3(-(-s)^2)+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)(r^2+2s+1)+1}$$

where

$\Omega_1=z^{n+3}(-1)(-s)^2W_{2n}+z^{n+2}((s-r)W_{2n+2}-((r+1)(s-r)+(-s))W_{2n+1}-(r+1)(-s)W_{2n})+z^{n+1}(-1)W_{2n+2}+z^2(-(s-r)W_2+((-s)+(r+1)(s-r))W_1+((s-r)^2-(r+1)(-s))W_0)+z(W_2-((r+1)^2+2(s-r))W_0)+W_0$.

(ii) *If* $z^3(-(-s)^2)+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)(r^2+2s+1)+1=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_2}{3z^2(-(-s)^2)+2z(-2(r+1)(-s)+(s-r)^2)+(-1)(r^2+2s+1)}$$

where

$\Omega_2=(n+3)z^{n+2}(-1)(-s)^2W_{2n}+(n+2)z^{n+1}((s-r)W_{2n+2}-((r+1)(s-r)+(-s))W_{2n+1}-(r+1)(-s)W_{2n})+(n+1)z^n(-1)W_{2n+2}+2z(-(s-r)W_2+((-s)+(r+1)(s-r))W_1+((s-r)^2-(r+1)(-s))W_0)+(W_2-((r+1)^2+2(s-r))W_0)$.

(iii) *If* $z^3(-(-s)^2)+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)(r^2+2s+1)+1=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$*, i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_3}{6z(-(-s)^2)+2(-2(r+1)(-s)+(s-r)^2)}$$

where

$\Omega_3 = (n+3)(n+2)z^{n+1}(-1)(-s)^2W_{2n}+(n+2)(n+1)z^n((s-r)W_{2n+2}-((r+1)(s-r)+(-s))W_{2n+1}-(r+1)(-s)W_{2n})+(n+1)nz^{n-1}(-1)W_{2n+2}+2(-(s-r)W_2+((-s)+(r+1)(s-r))W_1+((s-r)^2-(r+1)(-s))W_0)$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_4}{3z^2(-(-s)^2)+2z(-2(r+1)(-s)+(s-r)^2)+(-1)(r^2+2s+1)}$$

where

$\Omega_4 = (n+3)z^{n+2}(-1)(-s)^2W_{2n}+(n+2)z^{n+1}((s-r)W_{2n+2}-((r+1)(s-r)+(-s))W_{2n+1}-(r+1)(-s)W_{2n})+(n+1)z^n(-1)W_{2n+2}+2z(-(s-r)W_2+((-s)+(r+1)(s-r))W_1+((s-r)^2-(r+1)(-s))W_0)+(W_2-((r+1)^2+2(s-r))W_0)$.

(iv) *If* $z^3(-(-s)^2)+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)(r^2+2s+1)+1 = u(z-a)^3 = 0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$*, i.e.,* $z=a$*, then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_5}{6(-(-s)^2)}$$

where

$\Omega_5 = (n+3)(n+2)(n+1)z^n(-1)(-s)^2W_{2n}+(n+2)(n+1)nz^{n-1}((s-r)W_{2n+2}-((r+1)(s-r)+(-s))W_{2n+1}-(r+1)(-s)W_{2n})+(n+1)n(n-1)z^{n-2}(-1)W_{2n+2}$.

(c) $(m=2, j=1)$.

(i) *If* $z^3(-(-s)^2)+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)(r^2+2s+1)+1\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_1}{z^3(-(-s)^2)+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)(r^2+2s+1)+1}$$

where

$\Omega_1 = z^{n+3}(-(-s)^2W_{2n+1})+z^{n+2}(-(-s)W_{2n+2}+((s-r)^2-(r+1)(-s))W_{2n+1}+(s-r)(-s)W_{2n})+z^{n+1}(-1)((r+1)W_{2n+2}+(s-r)W_{2n+1}+(-s)W_{2n})+z^2(-s)(W_2-(r+1)W_1-(s-r)W_0)+z((r+1)W_2-((r+1)^2+(s-r))W_1+(-s)W_0)+W_1$.

(ii) *If* $z^3(-(-s)^2)+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)(r^2+2s+1)+1 = u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_2}{3z^2(-(-s)^2)+2z(-2(r+1)(-s)+(s-r)^2)+(-1)(r^2+2s+1)}$$

where

$\Omega_2 = (n+3)z^{n+2}(-(-s)^2W_{2n+1})+(n+2)z^{n+1}(-(-s)W_{2n+2}+((s-r)^2-(r+1)(-s))W_{2n+1}+(s-r)(-s)W_{2n})+(n+1)z^n(-1)((r+1)W_{2n+2}+(s-r)W_{2n+1}+(-s)W_{2n})+2z(-s)(W_2-(r+1)W_1-(s-r)W_0)+((r+1)W_2-((r+1)^2+(s-r))W_1+(-s)W_0)$.

(iii) *If* $z^3(-(-s)^2)+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)(r^2+2s+1)+1 = u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$*, i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_3}{6z(-(-s)^2)+2(-2(r+1)(-s)+(s-r)^2)}$$

where

$\Omega_3 = (n+3)(n+2)z^{n+1}(-(-s)^2W_{2n+1}) + (n+2)(n+1)z^n(-(-s)W_{2n+2} + ((s-r)^2 - (r+1)(-s))W_{2n+1} + (s-r)(-s)W_{2n}) + (n+1)nz^{n-1}(-1)((r+1)W_{2n+2} + (s-r)W_{2n+1} + (-s)W_{2n}) + 2(-s)(W_2 - (r+1)W_1 - (s-r)W_0)$

and for $z = b$ we get

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_4}{3z^2(-(-s)^2) + 2z(-2(r+1)(-s) + (s-r)^2) + (-1)(r^2+2s+1)}$$

where

$\Omega_4 = (n+3)z^{n+2}(-(-s)^2W_{2n+1}) + (n+2)z^{n+1}(-(-s)W_{2n+2} + ((s-r)^2 - (r+1)(-s))W_{2n+1} + (s-r)(-s)W_{2n}) + (n+1)z^n(-1)((r+1)W_{2n+2} + (s-r)W_{2n+1} + (-s)W_{2n}) + 2z(-s)(W_2 - (r+1)W_1 - (s-r)W_0) + ((r+1)W_2 - ((r+1)^2 + (s-r))W_1 + (-s)W_0)$.

(iv) *If $z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2+2s+1) + 1 = u(z-a)^3 = 0$ for some $u, a \in \mathbb{C}$ with $u \neq 0$, i.e., $z = a$, then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_5}{6(-(-s)^2)}$$

where

$\Omega_5 = (n+3)(n+2)(n+1)z^n(-(-s)^2W_{2n+1}) + (n+2)(n+1)nz^{n-1}(-(-s)W_{2n+2} + ((s-r)^2 - (r+1)(-s))W_{2n+1} + (s-r)(-s)W_{2n}) + (n+1)n(n-1)z^{n-2}(-1)((r+1)W_{2n+2} + (s-r)W_{2n+1} + (-s)W_{2n})$.

8.2 The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of Generalized Horadam-Leonardo Polynomials (in Terms of Generalized Horadam-Leonardo Polynomials and (r,s)-Horadam-Leonardo Polynomials)

The sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Horadam-Leonardo polynomials (in terms of elements of the sequence of generalized Horadam-Leonardo polynomials and (r,s)-Horadam-Leonardo polynomials) can be given given as follows.

Theorem 8.3. *For all $m, j \in \mathbb{Z}$, one get the following summing formulas.*

(a) *If $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 \neq 0$ then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4} = \frac{\Theta_G(z)}{\Gamma_G(z)}$$

where

$\Theta_G(z) = z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6,$

$z^{n+3}\Theta_1 = z^{n+3}((-W_jG_{m+2}^2 + (-W_{j+2} + (s-r)W_j)G_{m+1}^2 - (-s)W_{j+1}G_m^2 + (W_{j+1} + (r+1)W_j)G_{m+2}G_{m+1} + (W_{j+2} - (r+1)W_{j+1})G_{m+2}G_m + (-(s-r)W_{j+1} + (-s)W_j)G_{m+1}G_m)G_{m+mn+2} + ((-W_{j+1} + (r+1)W_j)G_{m+2}^2 - ((-s) + (r+1)(s-r))W_jG_{m+1}^2 + (-s)(-W_{j+2} + (r+1)W_{j+1})G_m^2 + (W_{j+2} - (r+1)^2W_j)G_{m+2}G_{m+1} + (-(r+1)W_{j+2} + ((r+1)^2 + (s-r))W_{j+1} + (-s)W_j)G_{m+2}G_m + (-(s-r)W_{j+2} + ((-s) + (r+1)(s-r))W_{j+1} - (r+1)(-s)W_j)G_{m+1}G_m)G_{m+mn+1} + ((-W_{j+2} + (r+1)W_{j+1} + (s-r)W_j)G_{m+2}^2 + ((s-r)W_{j+2} - ((-s) + (r+1)(s-r))W_{j+1} - (s-r)^2W_j)G_{m+1}^2 - W_j(-s)^2G_m^2 + (W_{j+2}(r+1) - (r+1)^2W_{j+1} + ((-s) - (r+1)(s-r))W_j)G_{m+2}G_{m+1} + (-s)(W_{j+1} - (r+1)W_j)G_{m+2}G_m + (-s)(W_{j+2} - (r+1)W_{j+1} - 2(s-r)W_j)G_{m+1}G_m)G_{m+mn}),$

$z^{n+2}\Theta_2 = z^{n+2}((((r+1)W_j - W_{j+1})G_{m+2} + (2W_{j+2} - (r+1)W_{j+1} - ((r+1)^2 + 2(s-r))W_j)G_{m+1} + (-(r+1)W_{j+2} + ((r+1)^2 + (s-r))W_{j+1} - (-s)W_j)G_m)G_{m+mn+2} + ((2(r+1)W_{j+1} - W_{j+2} - (r+$

$1)^2W_j)G_{m+2}+(-(r+1)W_{j+2}+((r+1)^3+2(-s)+2(r+1)(s-r))W_j)G_{m+1}+(((r+1)^2+(s-r))W_{j+2}-((r+1)^3+2(r+1)(s-r)+(-s))W_{j+1})G_m)G_{m+mn+1}+(((r+1)W_{j+2}-(r+1)^2W_{j+1}-((r+1)(s-r)+(-s))W_j)G_{m+2}+(-((r+1)^2+2(s-r))W_{j+2}+((r+1)^3+2(r+1)(s-r)+2(-s))W_{j+1}+((r+1)^2(s-r)+2(s-r)^2-(r+1)(-s))W_j)G_{m+1}+2(s-r)(-s)W_jG_m-(-s)G_mW_{j+2}+(r+1)^2(-s)W_jG_m)G_{m+mn}),$

$z^{n+1}\Theta_3=z^{n+1}((-W_{j+2}+(r+1)W_{j+1}+(s-r)W_j)G_{m+mn+2}+((r+1)W_{j+2}-(r+1)^2W_{j+1}-((r+1)(s-r)+(-s))W_j)G_{m+mn+1}+((s-r)W_{j+2}-((r+1)(s-r)+(-s))W_{j+1}-((s-r)^2-(r+1)(-s))W_j)G_{m+mn}),$

$z^2\Theta_4=z^2(G_{m+2}^2W_{j+1}+((r+1)W_{j+2}+(-s)W_j)G_{m+1}^2+(-s)G_m^2W_{j+2}-((r+1)W_{j+1}+W_{j+2})G_{m+1}G_{m+2}-((s-r)W_{j+1}+(-s)W_j)G_mG_{m+2}+((s-r)W_{j+2}-(-s)W_{j+1})G_mG_{m+1}),$

$z\Theta_5=z((W_{j+2}-(r+1)W_{j+1})G_{m+2}+(-(r+1)W_{j+2}+(r+1)^2W_{j+1}-2(-s)W_j)G_{m+1}+(-(s-r)W_{j+2}+((-s)W_{j+1}+(r+1)(s-r)W_{j+1})+(r+1)(-s)W_j)G_m),$

$\Theta_6=(-s)W_j,$

and

$\Gamma_G(z)=\Gamma_1(z)+\Gamma_2(z)+\Gamma_3(z)+\Gamma_4(z),$

$z^3\Gamma_1=z^3(-(-s)^{m+1}),$

$z^2\Gamma_2=z^2((r+1)G_{m+2}^2+((r+1)^3+2(r+1)(s-r)+3(-s))G_{m+1}^2+((r+1)^2(-s)+2(s-r)(-s))G_m^2-2((r+1)^2+(s-r))G_{m+1}G_{m+2}-((r+1)(s-r)+3(-s))G_mG_{m+2}+((r+1)^2(s-r)+2(s-r)^2-(r+1)(-s))G_mG_{m+1}),$

$z\Gamma_3=z((s-r)G_{m+2}-((r+1)(s-r)+3(-s))G_{m+1}+(2(r+1)(-s)-(s-r)^2)G_m),$

$\Gamma_4=(-s).$

(b) *If* $z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^kW_{mk+j}=\frac{(n+3)z^{n+2}\Theta_1+(n+2)z^{n+1}\Theta_2+(n+1)z^n\Theta_3+2z\Theta_4+\Theta_5}{3z^2\Gamma_1+2z\Gamma_2+\Gamma_3}.$$

(c) *If* $z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$*, i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^kW_{mk+j}=\frac{(n+3)(n+2)z^{n+1}\Theta_1+(n+2)(n+1)z^n\Theta_2+(n+1)nz^{n-1}\Theta_3+2\Theta_4}{6z\Gamma_1+2\Gamma_2},$$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^kW_{mk+j}=\frac{(n+3)z^{n+2}\Theta_1+(n+2)z^{n+1}\Theta_2+(n+1)z^n\Theta_3+2z\Theta_4+\Theta_5}{3z^2\Gamma_1+2z\Gamma_2+\Gamma_3}.$$

(d) *If* $z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$*, i.e.,* $z=a$*, then*

$$\sum_{k=0}^{n} z^kW_{mk+j}=\frac{(n+3)(n+2)(n+1)z^n\Theta_1+(n+2)(n+1)nz^{n-1}\Theta_2+(n+1)n(n-1)z^{n-2}\Theta_3}{6\Gamma_1}.$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Theorem 63]. □

Some special cases of the last Theorem can be given as follows.

Theorem 8.4. *One get the following sum formulas.*

(a) $(m=1,j=0)$.

(i) *If* $z^3(-1)(-s)+z^2(-(s-r))+z(-(r+1))+1\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_1}{z^3(-1)(-s)+z^2(-(s-r))+z(-(r+1))+1}$$

where

$\Omega_1 = z^{n+3}((-W_2+(r+1)W_1+(s-r)W_0)G_{n+2}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_{n+1}+((s-r)W_2-((r+1)(s-r)+(-s))W_1+((r+1)(-s)-(s-r)^2)W_0)G_n)+z^{n+2}((-W_1+(r+1)W_0)G_{n+2}-(W_2-2(r+1)W_1+(r+1)^2W_0)G_{n+1}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_n)+z^{n+1}(-W_0G_{n+2}+(-W_1+(r+1)W_0)G_{n+1}+(-W_2+(r+1)W_1+(s-r)W_0)G_n)+z^2(W_2-(r+1)W_1-(s-r)W_0)+z(W_1-(r+1)W_0)+W_0.$

(ii) *If* $z^3(-1)(-s)+z^2(-(s-r))+z(-(r+1))+1=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_2}{3z^2(-1)(-s)+2z(-(s-r))+(-(r+1))}$$

where

$\Omega_2 = (n+3)z^{n+2}((-W_2+(r+1)W_1+(s-r)W_0)G_{n+2}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_{n+1}+((s-r)W_2-((r+1)(s-r)+(-s))W_1+((r+1)(-s)-(s-r)^2)W_0)G_n)+(n+2)z^{n+1}((-W_1+(r+1)W_0)G_{n+2}-(W_2-2(r+1)W_1+(r+1)^2W_0)G_{n+1}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_n)+(n+1)z^n(-W_0G_{n+2}+(-W_1+(r+1)W_0)G_{n+1}+(-W_2+(r+1)W_1+(s-r)W_0)G_n)+2z(W_2-(r+1)W_1-(s-r)W_0)+(W_1-(r+1)W_0).$

(iii) *If* $z^3(-1)(-s)+z^2(-(s-r))+z(-(r+1))+1=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$*, i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_3}{6z(-1)(-s)+2(-(s-r))}$$

where

$\Omega_3 = (n+3)(n+2)z^{n+1}((-W_2+(r+1)W_1+(s-r)W_0)G_{n+2}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_{n+1}+((s-r)W_2-((r+1)(s-r)+(-s))W_1+((r+1)(-s)-(s-r)^2)W_0)G_n)+(n+2)(n+1)z^n((-W_1+(r+1)W_0)G_{n+2}-(W_2-2(r+1)W_1+(r+1)^2W_0)G_{n+1}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_n)+(n+1)nz^{n-1}(-W_0G_{n+2}+(-W_1+(r+1)W_0)G_{n+1}+(-W_2+(r+1)W_1+(s-r)W_0)G_n)+2(W_2-(r+1)W_1-(s-r)W_0)$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_4}{3z^2(-1)(-s)+2z(-(s-r))+(-(r+1))}$$

where

$\Omega_4 = (n+3)z^{n+2}((-W_2+(r+1)W_1+(s-r)W_0)G_{n+2}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_{n+1}+((s-r)W_2-((r+1)(s-r)+(-s))W_1+((r+1)(-s)-(s-r)^2)W_0)G_n)+(n+2)z^{n+1}((-W_1+(r+1)W_0)G_{n+2}-(W_2-2(r+1)W_1+(r+1)^2W_0)G_{n+1}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_n)+(n+1)z^n(-W_0G_{n+2}+(-W_1+(r+1)W_0)G_{n+1}+(-W_2+(r+1)W_1+(s-r)W_0)G_n)+2z(W_2-(r+1)W_1-(s-r)W_0)+(W_1-(r+1)W_0)$

(iv) *If* $z^3(-1)(-s)+z^2(-(s-r))+z(-(r+1))+1=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$*, i.e.,* $z=a$*, then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_5}{6(-1)(-s)}$$

where

$\Omega_5=(n+3)(n+2)(n+1)z^n((-W_2+(r+1)W_1+(s-r)W_0)G_{n+2}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_{n+1}+((s-r)W_2-((r+1)(s-r)+(-s))W_1+((r+1)(-s)-(s-r)^2)W_0)G_n)+(n+2)(n+1)nz^{n-1}((-W_1+(r+1)W_0)G_{n+2}-(W_2-2(r+1)W_1+(r+1)^2W_0)G_{n+1}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_n)+(n+1)n(n-1)z^{n-2}(-W_0G_{n+2}+(-W_1+(r+1)W_0)G_{n+1}+(-W_2+(r+1)W_1+(s-r)W_0)G_n)$

(b) $(m=2,j=0)$.

(i) *If* $z^3(-1)(-s)^2+z^2((s-r)^2-2(r+1)(-s))+z(-((r+1)^2+2(s-r)))+1\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k}=\frac{\Omega_1}{z^3(-1)(-s)^2+z^2((s-r)^2-2(r+1)(-s))+z(-((r+1)^2+2(s-r)))+1}$$

where

$\Omega_1=z^{n+3}(-s)((-W_2+(r+1)W_1+(s-r)W_0)G_{2n+2}+((r+1)W_2-(r+1)^2W_1-((-s)+(r+1)(s-r))W_0)G_{2n+1}+((s-r)W_2-((-s)+(r+1)(s-r))W_1+((r+1)(-s)-(s-r)^2)W_0)G_{2n})+z^{n+2}((-(r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n+2}+(((r+1)^2+(s-r))W_2-((r+1)^3+2(r+1)(s-r)+(-s))W_1)G_{2n+1}+(-s)(-W_2+2(s-r)W_0+(r+1)^2W_0)G_{2n})+z^{n+1}(-W_1G_{2n+2}+((r+1)W_1-W_2)G_{2n+1}-(-s)W_0G_{2n})+z^2(-(s-r)W_2+((r+1)(s-r)+(-s))W_1+((s-r)^2-(r+1)(-s))W_0)+z(W_2-((r+1)^2+2(s-r))W_0)+W_0$.

(ii) *If* $z^3(-1)(-s)^2+z^2((s-r)^2-2(r+1)(-s))+z(-((r+1)^2+2(s-r)))+1=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_{2k}=\frac{\Omega_2}{3z^2(-1)(-s)^2+2z((s-r)^2-2(r+1)(-s))+(-((r+1)^2+2(s-r)))}$$

where

$\Omega_2=(n+3)z^{n+2}(-s)((-W_2+(r+1)W_1+(s-r)W_0)G_{2n+2}+((r+1)W_2-(r+1)^2W_1-((-s)+(r+1)(s-r))W_0)G_{2n+1}+((s-r)W_2-((-s)+(r+1)(s-r))W_1+((r+1)(-s)-(s-r)^2)W_0)G_{2n})+(n+2)z^{n+1}((-(r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n+2}+(((r+1)^2+(s-r))W_2-((r+1)^3+2(r+1)(s-r)+(-s))W_1)G_{2n+1}+(-s)(-W_2+2(s-r)W_0+(r+1)^2W_0)G_{2n})+(n+1)z^n(-W_1G_{2n+2}+((r+1)W_1-W_2)G_{2n+1}-(-s)W_0G_{2n})+2z(-(s-r)W_2+((r+1)(s-r)+(-s))W_1+((s-r)^2-(r+1)(-s))W_0)+(W_2-((r+1)^2+2(s-r))W_0)$.

(iii) *If* $z^3(-1)(-s)^2+z^2((s-r)^2-2(r+1)(-s))+z(-((r+1)^2+2(s-r)))+1=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$*, i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k}=\frac{\Omega_3}{6z(-1)(-s)^2+2((s-r)^2-2(r+1)(-s))}$$

where

$\Omega_3=(n+3)(n+2)z^{n+1}(-s)((-W_2+(r+1)W_1+(s-r)W_0)G_{2n+2}+((r+1)W_2-(r+1)^2W_1-((-s)+(r+1)(s-r))W_0)G_{2n+1}+((s-r)W_2-((-s)+(r+1)(s-r))W_1+((r+1)(-s)-(s-r)^2)W_0)G_{2n})+(n+2)(n+1)z^n((-(r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n+2}+(((r+1)^2+(s-r))W_2-((r+1)^3+2(r+1)(s-r)+(-s))W_1)G_{2n+1}+(-s)(-W_2+2(s-r)W_0+(r+1)^2W_0)G_{2n})+(n+1)nz^{n-1}(-W_1G_{2n+2}+((r+1)W_1-W_2)G_{2n+1}-(-s)W_0G_{2n})+2(-(s-r)W_2+((r+1)(s-r)+(-s))W_1+((s-r)^2-(r+1)(-s))W_0)$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k}=\frac{\Omega_4}{3z^2(-1)(-s)^2+2z((s-r)^2-2(r+1)(-s))+(-((r+1)^2+2(s-r)))}$$

where

$\Omega_4 = (n+3)z^{n+2}(-s)((-W_2+(r+1)W_1+(s-r)W_0)G_{2n+2}+((r+1)W_2-(r+1)^2W_1-((-s)+(r+1)(s-r))W_0)G_{2n+1}+((s-r)W_2-((-s)+(r+1)(s-r))W_1+((r+1)(-s)-(s-r)^2)W_0)G_{2n})+(n+2)z^{n+1}((-(r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n+2}+(((r+1)^2+(s-r))W_2-((r+1)^3+2(r+1)(s-r)+(-s))W_1)G_{2n+1}+(-s)(-W_2+2(s-r)W_0+(r+1)^2W_0)G_{2n})+(n+1)z^n(-W_1G_{2n+2}+((r+1)W_1-W_2)G_{2n+1}-(-s)W_0G_{2n})+2z(-(s-r)W_2+((r+1)(s-r)+(-s))W_1+((s-r)^2-(r+1)(-s))W_0)+(W_2-((r+1)^2+2(s-r))W_0).$

(iv) *If* $z^3(-1)(-s)^2+z^2((s-r)^2-2(r+1)(-s))+z(-((r+1)^2+2(s-r)))+1=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$*, i.e.,* $z=a$*, then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_5}{6(-1)(-s)^2}$$

where

$\Omega_5 = (n+3)(n+2)(n+1)z^n(-s)((-W_2+(r+1)W_1+(s-r)W_0)G_{2n+2}+((r+1)W_2-(r+1)^2W_1-((-s)+(r+1)(s-r))W_0)G_{2n+1}+((s-r)W_2-((-s)+(r+1)(s-r))W_1+((r+1)(-s)-(s-r)^2)W_0)G_{2n})+(n+2)(n+1)nz^{n-1}((-(r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n+2}+(((r+1)^2+(s-r))W_2-((r+1)^3+2(r+1)(s-r)+(-s))W_1)G_{2n+1}+(-s)(-W_2+2(s-r)W_0+(r+1)^2W_0)G_{2n})+(n+1)n(n-1)z^{n-2}(-W_1G_{2n+2}+((r+1)W_1-W_2)G_{2n+1}-(-s)W_0G_{2n}).$

(c) $(m=2, j=1)$.

(i) *If* $z^3(-1)(-s)^2+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)((r+1)^2+2(s-r))+1\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_1}{z^3(-1)(-s)^2+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)((r+1)^2+2(s-r))+1}$$

where

$\Omega_1 = z^{n+3}(-s)^2(-W_0G_{2n+2}+((r+1)W_0-W_1)G_{2n+1}+(-W_2+(r+1)W_1+(s-r)W_0)G_{2n})+z^{n+2}(((s-r)W_2-((r+1)(s-r)+(-s))W_1-(r+1)(-s)W_0)G_{2n+2}+(-((r+1)(s-r)+(-s))W_2+(s-r)((r+1)^2+(s-r))W_1+(-s)((s-r)+(r+1)^2)W_0)G_{2n+1}+(-s)(-(r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n})+z^{n+1}(-W_2G_{2n+2}-((s-r)W_1+(-s)W_0)G_{2n+1}-(-s)W_1G_{2n})+z^2(-s)(W_2-(r+1)W_1-(s-r)W_0)+z((r+1)W_2-((r+1)^2+(s-r))W_1+(-s)W_0)+W_1.$

(ii) *If* $z^3(-1)(-s)^2+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)((r+1)^2+2(s-r))+1=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_2}{3z^2(-1)(-s)^2+2z(-2(r+1)(-s)+(s-r)^2)+(-1)((r+1)^2+2(s-r))}$$

where

$\Omega_2 = (n+3)z^{n+2}(-s)^2(-W_0G_{2n+2}+((r+1)W_0-W_1)G_{2n+1}+(-W_2+(r+1)W_1+(s-r)W_0)G_{2n})+(n+2)z^{n+1}(((s-r)W_2-((r+1)(s-r)+(-s))W_1-(r+1)(-s)W_0)G_{2n+2}+(-((r+1)(s-r)+(-s))W_2+(s-r)((r+1)^2+(s-r))W_1+(-s)((s-r)+(r+1)^2)W_0)G_{2n+1}+(-s)(-(r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n})+(n+1)z^n(-W_2G_{2n+2}-((s-r)W_1+(-s)W_0)G_{2n+1}-(-s)W_1G_{2n})+2z(-s)(W_2-(r+1)W_1-(s-r)W_0)+((r+1)W_2-((r+1)^2+(s-r))W_1+(-s)W_0).$

(iii) *If* $z^3(-1)(-s)^2+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)((r+1)^2+2(s-r))+1=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$*, i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_3}{6z(-1)(-s)^2+2(-2(r+1)(-s)+(s-r)^2)}$$

where

$\Omega_3 = (n+3)(n+2)z^{n+1}(-s)^2(-W_0G_{2n+2} + ((r+1)W_0 - W_1)G_{2n+1} + (-W_2 + (r+1)W_1 + (s-r)W_0)G_{2n}) + (n+2)(n+1)z^n(((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 - (r+1)(-s)W_0) G_{2n+2} + (-((r+1)(s-r) + (-s))W_2 + (s-r)((r+1)^2 + (s-r))W_1 + (-s)((s-r) + (r+1)^2)W_0) G_{2n+1} + (-s)(-(r+1)W_2 + ((r+1)^2 + (s-r))W_1 - (-s)W_0)G_{2n}) + (n+1)nz^{n-1}(-W_2G_{2n+2} - ((s-r)W_1 + (-s)W_0)G_{2n+1} - (-s)W_1G_{2n}) + 2(-s)(W_2 - (r+1)W_1 - (s-r)W_0)$

and for $z = b$ we get

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_4}{3z^2(-1)(-s)^2 + 2z(-2(r+1)(-s) + (s-r)^2) + (-1)((r+1)^2 + 2(s-r))}$$

where

$\Omega_4 = (n+3)z^{n+2}(-s)^2(-W_0G_{2n+2} + ((r+1)W_0 - W_1)G_{2n+1} + (-W_2 + (r+1)W_1 + (s-r)W_0)G_{2n}) + (n+2)z^{n+1}(((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 - (r+1)(-s)W_0)G_{2n+2} + (-((r+1)(s-r) + (-s))W_2 + (s-r)((r+1)^2 + (s-r))W_1 + (-s)((s-r) + (r+1)^2)W_0)G_{2n+1} + (-s)(-(r+1)W_2 + ((r+1)^2 + (s-r))W_1 - (-s)W_0)G_{2n}) + (n+1)z^n(-W_2G_{2n+2} - ((s-r)W_1 + (-s)W_0)G_{2n+1} - (-s)W_1 G_{2n}) + 2z(-s)(W_2 - (r+1)W_1 - (s-r)W_0) + ((r+1)W_2 - ((r+1)^2 + (s-r))W_1 + (-s)W_0).$

(iv) *If $z^3(-1)(-s)^2 + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)((r+1)^2 + 2(s-r)) + 1 = u(z-a)^3 = 0$ for some $u, a \in \mathbb{C}$ with $u \neq 0$, i.e., $z = a$, then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_5}{6(-1)(-s)^2}$$

where

$\Omega_5 = (n+3)(n+2)(n+1)z^n(-s)^2(-W_0G_{2n+2} + ((r+1)W_0 - W_1)G_{2n+1} + (-W_2 + (r+1)W_1 + (s-r)W_0)G_{2n}) + (n+2)(n+1)nz^{n-1}(((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 - (r+1)(-s)W_0) G_{2n+2} + (-((r+1)(s-r) + (-s))W_2 + (s-r)((r+1)^2 + (s-r))W_1 + (-s)((s-r) + (r+1)^2)W_0) G_{2n+1} + (-s)(-(r+1)W_2 + ((r+1)^2 + (s-r))W_1 - (-s)W_0)G_{2n}) + (n+1)n(n-1)z^{n-2}(-W_2G_{2n+2} - ((s-r)W_1 + (-s)W_0)G_{2n+1} - (-s)W_1G_{2n}).$

8.3 The Summing Formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of Generalized Horadam-Leonardo Polynomials (in Terms of Generalized Horadam-Leonardo Polynomials and (r,s)-Horadam-Leonardo-Lucas Polynomials)

The sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Horadam-Leonardo polynomials (in terms of elements of the sequence of generalized Horadam-Leonardo polynomials and (r,s)-Horadam-Leonardo-Lucas polynomials) can be given as follows.

Theorem 8.5. *For all $m, j \in \mathbb{Z}$, one get the following summing formulas.*

(a) *If $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 \neq 0$ then*

$$\begin{aligned}\sum_{k=0}^{n} z^k W_{mk+j} &= \frac{z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4} \\ &= \frac{\Theta_W(z)}{\Gamma_W(z)}\end{aligned}$$

where

$\Theta_W(z) = z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6,$

$z^{n+3}\Theta_1 = z^{n+3}((-W_jH_{m+2}^2 + (-W_{j+2} + (s-r)W_j)H_{m+1}^2 - (-s)W_{j+1}H_m^2 + (W_{j+1} + (r+1)W_j)H_{m+2}H_{m+1} + (W_{j+2} - (r+1)W_{j+1})H_{m+2}H_m + (-(s-r)W_{j+1} + (-s)W_j)H_{m+1}H_m)H_{m+mn+2} + ((-W_{j+1} + (r+1)W_j) H_{m+2}^2 - ((-s) + (r+1)(s-r))W_jH_{m+1}^2 + (-s)(-W_{j+2} + (r+1)W_{j+1})H_m^2 + (W_{j+2} - (r+1)^2W_j)H_{m+2}H_{m+1} +$

$(-(r+1)W_{j+2}+((r+1)^2+(s-r))W_{j+1}+(-s)W_j)H_{m+2}H_m+(-(s-r)W_{j+2}+((r+1)(s-r)+(-s))W_{j+1}-(r+1)(-s)W_j)H_{m+1}H_m)H_{m+mn+1}+((-W_{j+2}+(r+1)W_{j+1}+(s-r)W_j)H_{m+2}^2+((s-r)W_{j+2}-((r+1)(s-r)+(-s))W_{j+1}-(s-r)^2W_j)H_{m+1}^2-W_j(-s)^2H_m^2+((r+1)W_{j+2}-(r+1)^2W_{j+1}+((-s)-(r+1)(s-r))W_j)H_{m+2}H_{m+1}+(-s)(W_{j+1}-(r+1)W_j)H_{m+2}H_m+(-s)(W_{j+2}-(r+1)W_{j+1}-2(s-r)W_j)H_{m+1}H_m)H_{m+mn}),$

$z^{n+2}\Theta_2 = z^{n+2}(((-3W_{j+2}+2(r+1)W_{j+1}+((r+1)^2+4(s-r))W_j)H_{m+2}+(2(r+1)W_{j+2}+((s-r)-(r+1)^2)W_{j+1}-((r+1)^3+4(r+1)(s-r)+3(-s))W_j)H_{m+1}+(-((r+1)^2+2(s-r))W_{j+2}+((r+1)^3+3(r+1)(s-r)+6(-s))W_{j+1}-(r+1)(-s)W_j)H_m)H_{m+mn+2}+((2(r+1)W_{j+2}-((r+1)^2-(s-r))W_{j+1}-((r+1)^3+3(-s)+4(r+1)(s-r))W_j)H_{m+2}+(-((r+1)^2-(s-r))W_{j+2}-3((-s)+(r+1)(s-r))W_{j+1}+(r+1)((r+1)^3+4(r+1)(s-r)+5(-s))W_j)H_{m+1}+(((r+1)^3+3(r+1)(s-r)+6(-s))W_{j+2}-((r+1)^4+4(r+1)^2(s-r)+2(s-r)^2+7(r+1)(-s))W_{j+1}-2(s-r)(-s)W_j)H_m)H_{m+mn+1}+((((r+1)^2+4(s-r))W_{j+2}-((r+1)^3+4(r+1)(s-r)+3(-s))W_{j+1}-((r+1)^2(s-r)-2(r+1)(-s)+4(s-r)^2)W_j)H_{m+2}+(-((r+1)^3+4(r+1)(s-r)+3(-s))W_{j+2}+((r+1)^4+4(r+1)^2(s-r)+5(r+1)(-s))W_{j+1}+((r+1)^3(s-r)-(r+1)^2(-s)+4(r+1)(s-r)^2+4(s-r)(-s))W_j)H_{m+1}+(-s)(-(r+1)W_{j+2}-2(s-r)W_{j+1}+((r+1)^3+4(r+1)(s-r)+6(-s))W_j)H_m)H_{m+mn}),$

$z^{n+1}\Theta_3 = z^{n+1}((2(3(s-r)+(r+1)^2)W_{j+2}-(2(r+1)^3+7(r+1)(s-r)+9(-s))W_{j+1}-((r+1)^2(s-r)-3(r+1)(-s)+4(s-r)^2)W_j)H_{m+mn+2}+(-(2(r+1)^3+9(-s)+7(r+1)(s-r))W_{j+2}+2((r+1)^4+4(r+1)^2(s-r)+6(r+1)(-s)+(s-r)^2)W_{j+1}+((r+1)^3(s-r)-(r+1)^2(-s)+4(r+1)(s-r)^2+6(s-r)(-s))W_j)H_{m+mn+1}+(-((r+1)^2(s-r)-3(r+1)(-s)+4(s-r)^2)W_{j+2}+((r+1)^3(s-r)-(r+1)^2(-s)+4(r+1)(s-r)^2+6(s-r)(-s))W_{j+1}+(-2(r+1)^3(-s)+(r+1)^2(s-r)^2+4(s-r)^3-9(-s)^2-10(r+1)(s-r)(-s))W_j)H_{m+mn}),$

$z^2\Theta_4 = z^2((3W_{j+2}-2(r+1)W_{j+1}-(s-r)W_j)H_{m+2}^2+(((r+1)^2-(s-r))W_{j+2}+3((r+1)(s-r)+(-s))W_{j+1}+((s-r)^2+(r+1)(-s))W_j)H_{m+1}^2+(-s)((r+1)W_{j+2}+2(s-r)W_{j+1}+3(-s)W_j)H_m^2+(-4(r+1)W_{j+2}+2((r+1)^2-(s-r))W_{j+1}+((r+1)(s-r)-3(-s))W_j)H_{m+2}H_{m+1}+(-2(s-r)W_{j+2}+((r+1)(s-r)-3(-s))W_{j+1}+2(r+1)(-s)W_j)H_{m+2}H_m+(((r+1)(s-r)-3(-s))W_{j+2}+2((s-r)^2+(r+1)(-s))W_{j+1}+4(s-r)(-s)W_j)H_{m+1}H_m),$

$z\Theta_5 = z((-2((r+1)^2+3(s-r))W_{j+2}+(2(r+1)^3+7(r+1)(s-r)+9(-s))W_{j+1}+((r+1)^2(s-r)-3(r+1)(-s)+4(s-r)^2)W_j)H_{m+2}+((2(r+1)^3+7(r+1)(s-r)+9(-s))W_{j+2}-2((r+1)^4+4(r+1)^2(s-r)+6(r+1)(-s)+(s-r)^2)W_{j+1}-((r+1)^3(s-r)+4(r+1)(s-r)^2-(r+1)^2(-s)+6(s-r)(-s))W_j)H_{m+1}+(((r+1)^2(s-r)+4(s-r)^2-3(r+1)(-s))W_{j+2}-(4(r+1)(s-r)^2+(r+1)^3(s-r)-(r+1)^2(-s)+6(s-r)(-s))W_{j+1}-2(-s)((r+1)^3+4(r+1)(s-r)+9(-s))W_j)H_m),$

$\Theta_6 = (4(r+1)^3(-s)-(r+1)^2(s-r)^2-4(s-r)^3+27(-s)^2+18(r+1)(s-r)(-s))W_j,$

and

$\Gamma_W(z) = \Gamma_1(z)+\Gamma_2(z)+\Gamma_3(z)+\Gamma_4(z),$

$z^3\Gamma_1 = z^3(-(-s)^m(4(r+1)^3(-s)-(r+1)^2(s-r)^2+18(r+1)(s-r)(-s)-4(s-r)^3+27(-s)^2)),$

$z^2\Gamma_2 = z^2(((r+1)^2+3(s-r))H_{m+2}^2+((r+1)^4+4(r+1)^2(s-r)+(s-r)^2+6(r+1)(-s))H_{m+1}^2+(-s)((r+1)^3+4(r+1)(s-r)+9(-s))H_m^2-(2(r+1)^3+7(r+1)(s-r)+9(-s))H_{m+2}H_{m+1}-((r+1)^2(s-r)+4(s-r)^2-3(r+1)(-s))H_{m+2}H_m+(r+1)(s-r)((r+1)^2+4(s-r))H_{m+1}H_m-(-s)((r+1)^2-6(s-r))H_{m+1}H_m),$

$z\Gamma_3 = z(-4(r+1)^3(-s)+(r+1)^2(s-r)^2+4(s-r)^3-27(-s)^2-18(r+1)(s-r)(-s))H_m,$

$\Gamma_4 = 4(r+1)^3(-s)-(r+1)^2(s-r)^2+18(r+1)(s-r)(-s)-4(s-r)^3+27(-s)^2.$

(b) *If* $z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4 = u(z-a)(z-b)(z-c) = 0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1+(n+2)z^{n+1}\Theta_2+(n+1)z^n\Theta_3+2z\Theta_4+\Theta_5}{3z^2\Gamma_1+2z\Gamma_2+\Gamma_3}.$$

(c) *If* $z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4 = u(z-a)^2(z-b) = 0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$*, i.e.,* $z=a$ *or* $z=b$

then for $z = a$ we get

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)(n+2)z^{n+1}\Theta_1 + (n+2)(n+1)z^n\Theta_2 + (n+1)nz^{n-1}\Theta_3 + 2\Theta_4}{6z\Gamma_1 + 2\Gamma_2},$$

and for $z = b$ we get

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(d) *If $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^3 = 0$ for some $u, a \in \mathbb{C}$ with $u \neq 0$, i.e., $z = a$, then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)(n+2)(n+1)z^n\Theta_1 + (n+2)(n+1)nz^{n-1}\Theta_2 + (n+1)n(n-1)z^{n-2}\Theta_3}{6\Gamma_1}.$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Theorem 65]. □

9 GENERALIZED HORADAM-LEONARDO POLYNOMIALS: GENERATING FUNCTION

Now, we give generating function of the sequence W_{mn+j} and some special cases of its.

9.1 Generating Function of Generalized Horadam-Leonardo Polynomials via Generalized Horadam-Leonardo Polynomials

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_{mn+j} z^n$ of the sequence W_{mn+j} (in terms of elements of the sequence of generalized Horadam-Leonardo polynomials).

Lemma 9.1. *Suppose that $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, 1\}$. Suppose that $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j} z^n$ is the generating function of the generalized Horadam-Leonardo polynomials $\{W_{mn+j}\}$. Then, $\sum_{n=0}^{\infty} W_{mn+j} z^n$ is given by*

$$\sum_{n=0}^{\infty} W_{mn+j} z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 8.1 (a))

$z^2\Theta_4 = z^2((W_0W_{j+2} + (W_1 - (r+1)W_0)W_{j+1} + (W_2 - (r+1)W_1 - (s-r)W_0)W_j)W_{m+2}^2 + ((W_2 - (s-r)W_0)W_{j+2} + ((-s)W_0 + (r+1)(s-r)W_0)W_{j+1} + ((s-r)^2W_0 + ((r+1)(s-r) + (-s))W_1 - (s-r)W_2)W_j)W_{m+1}^2 + (-s)(W_1W_{j+2} + (W_2 - (r+1)W_1)W_{j+1} + (-s)W_0W_j)W_m^2 + (-(W_1 + (r+1)W_0)W_{j+2} + ((r+1)^2W_0 - W_2)W_{j+1} + (-(r+1)W_2 + (r+1)^2W_1 + ((r+1)(s-r) - (-s))W_0)W_j)W_{m+1}W_{m+2} + (((r+1)W_1 - W_2)W_{j+2} + ((r+1)W_2 - ((s-r) + (r+1)^2)W_1 - (-s)W_0)W_{j+1} + (-s)((r+1)W_0 - W_1)W_j)W_{m+2}W_m + (((s-r)W_1 - (-s)W_0)W_{j+2} + ((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 + (r+1)(-s)W_0)W_{j+1} + (-s)(-W_2 + (r+1)W_1 + 2(s-r)W_0)W_j)$ $W_{m+1}W_m),$

$z\Theta_5 = z((((W_1^2 - W_0W_2)W_{j+2} + ((-s)W_0^2 - W_1W_2 + (r+1)W_0W_2 + (s-r)W_0W_1)W_{j+1} + (-2W_2^2 - (r+1)^2W_1^2 - (-s)(r+1)W_0^2 + 3(r+1)W_1W_2 + 2(s-r)W_0W_2 + (2(-s) - (s-r)(r+1))W_0W_1)W_j)W_{m+2} + ((((-s)W_0^2 - W_1W_2 + (r+1)W_0W_2 + (s-r)W_0W_1)W_{j+2} + (W_2^2 - ((r+1)^2 + (s-r))W_0W_2 - ((r+1)(s-r) + (-s))W_0W_1)$ $W_{j+1} + ((r+1)W_2^2 - 2((-s) + (r+1)(s-r))W_1^2 - 2(s-r)(-s)W_0^2 + (2(s-r) - (r+1)^2)W_1W_2 + (2(-s) - (r+1)(s-r))W_0W_2 - (2(s-r)^2 + (r+1)(-s))W_0W_1)W_j)W_{m+1} + ((W_2^2 - (s-r)W_1^2 - (r+1)W_1W_2 -$

$(-s)W_0W_1)W_{j+2}+(-(r+1)W_2^2+((r+1)(s-r)+(-s))W_1^2+(r+1)^2W_1W_2-(-s)W_0W_2+(r+1)(-s)W_0W_1)$ $W_{j+1}+(-s)(-(r+1)W_1^2-2(-s)W_0^2+2W_1W_2-(r+1)W_0W_2-2(s-r)W_0W_1)W_j)W_m-(r+1)(-s)W_0^2W_{j+1}$ $W_{m+1}),$

$\Theta_6=(W_2^3+((-s)+(r+1)(s-r))W_1^3+(-s)^2W_0^3-2(r+1)W_1W_2^2-(s-r)W_0W_2^2+((r+1)^2-(s-r))W_1^2W_2+((s-r)^2+(r+1)(-s))W_0W_1^2+2(s-r)(-s)W_0^2W_1+(r+1)(-s)W_0^2W_2+((r+1)(s-r)-3(-s))$ $W_0W_1W_2)W_j,$

and

$z^3\Gamma_1=z^3(-(-s)^m(W_2^3+((-s)+(r+1)(s-r))W_1^3+(-s)^2W_0^3+((r+1)^2-(s-r))W_1^2W_2-2(r+1)W_1W_2^2-(s-r)W_0W_2^2+(r+1)(-s)W_0^2W_2+((s-r)^2+(r+1)(-s))W_0W_1^2+2(s-r)(-s)W_0^2W_1+((r+1)(s-r)-3(-s))W_0W_1W_2)),$

$z^2\Gamma_2=z^2((3W_2-2(r+1)W_1-(s-r)W_0)W_{m+2}^2+(((r+1)^2-(s-r))W_2+(3(r+1)(s-r)+3(-s))W_1+((s-r)^2+(r+1)(-s))W_0)W_{m+1}^2+(-s)((r+1)W_2+2(s-r)W_1+3(-s)W_0)W_m^2+(-4(r+1)W_2+2((r+1)^2-(s-r))W_1+((r+1)(s-r)-3(-s))W_0)W_{m+2}W_{m+1}+(-2(s-r)W_2+((r+1)(s-r)-3(-s))W_1+2(r+1)(-s)W_0)$ $W_{m+2}W_m+(((r+1)(s-r)-3(-s))W_2+2((s-r)^2+(r+1)(-s))W_1+4(s-r)(-s)W_0)W_{m+1}W_m),$

$z\Gamma_3=z((-3W_2^2+((s-r)-(r+1)^2)W_1^2-(-s)(r+1)W_0^2+4(r+1)W_1W_2+2(s-r)W_0W_2+(3(-s)-(s-r)(r+1))W_0W_1)W_{m+2}+(2(r+1)W_2^2-(3(r+1)(s-r)+3(-s))W_1^2-2(s-r)(-s)W_0^2+(2(s-r)-2(r+1)^2)W_1W_2+(3(-s)-(r+1)(s-r))W_0W_2-2((s-r)^2+(r+1)(-s))W_0W_1)W_{m+1}+((s-r)W_2^2-((s-r)^2+(r+1)(-s))W_1^2-3(-s)^2$ $W_0^2+(3(-s)-(r+1)(s-r))W_1W_2-2(r+1)(-s)W_0W_2-4(s-r)(-s)W_0W_1)W_m),$

$\Gamma_4=W_2^3+((-s)+(r+1)(s-r))W_1^3+(-s)^2W_0^3-2(r+1)W_1W_2^2+((r+1)^2-(s-r))W_1^2W_2-(s-r)W_0W_2^2+((s-r)^2+(r+1)(-s))W_0W_1^2+(r+1)(-s)W_0^2W_2+2(s-r)(-s)W_0^2W_1+((r+1)(s-r)-3(-s))W_0W_1W_2.$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Lemma 66]. □

Now, we consider special cases of the last Lemma.

Corollary 9.2. *The ordinary generating functions of the sequences* W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} *are given as follows:*

(a) $(m=1, j=0, |z|<\min\{|\alpha|^{-1}, |\beta|^{-1}, 1\})$.

$$\sum_{n=0}^{\infty}W_nz^n=\frac{z^2(W_2-(r+1)W_1-(s-r)W_0)+z(W_1-(r+1)W_0)+W_0}{z^3s+z^2(-1)(s-r)+z(-1)(r+1)+1}.$$

(b) $(m=2, j=0, |z|<\min\{|\alpha|^{-2}, |\beta|^{-2}, 1\})$.

$$\sum_{n=0}^{\infty}W_{2n}z^n=\frac{z^2(-(s-r)W_2-r(r-s+1)W_1+(r^2-rs+s^2+s)W_0)+z(W_2-(r^2+2s+1)W_0)+W_0}{z^3(-s^2)+z^2(r^2+s^2+2s)+z(-1)(r^2+2s+1)+1}.$$

(c) $(m=2, j=1, |z|<\min\{|\alpha|^{-2}, |\beta|^{-2}, 1\})$.

$$\sum_{n=0}^{\infty}W_{2n+1}z^n=\frac{z^2(-s)(W_2-(r+1)W_1-(s-r)W_0)+z((r+1)W_2-(r^2+r+s+1)W_1+(-s)W_0)+W_1}{z^3(-s^2)+z^2(r^2+s^2+2s)+z(-1)(r^2+2s+1)+1}.$$

(d) $(m=-1, j=0, |z|<\min\{|\alpha|, |\beta|, 1\})$.

$$\sum_{n=0}^{\infty}W_{-n}z^n=\frac{z^2W_1+z(W_2-(r+1)W_1)+(-s)W_0}{z^3(-1)+z^2(r+1)+z(s-r)+(-s)}.$$

(e) $(m=-2, j=0, |z|<\min\{|\alpha|^2, |\beta|^2, 1\})$.

$$\sum_{n=0}^{\infty}W_{-2n}z^n=\frac{z^2W_2+z(-(s-r)W_2-r(r-s+1)W_1+(r+1)(-s)W_0)+s^2W_0}{z^3(-1)+z^2(r^2+2s+1)+z(-1)(r^2+s^2+2s)+s^2}.$$

(f) $(m=-2, j=1, |z|<\min\{|\alpha|^2, |\beta|^2, 1\})$.

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{z^2((r+1)W_2+(s-r)W_1-sW_0)+z(-sW_2-(r^2+s^2+s-rs)W_1-s(r-s)W_0)+s^2W_1}{z^3(-1)+z^2(r^2+2s+1)+z(-1)(r^2+s^2+2s)+s^2}.$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Corollary 67.]. □

As particular examples (generating functions of (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials), from the last Lemma, we obtain the following results

Corollary 9.3. *Suppose that* $|z|<\min\{|\alpha|^{-m}, |\beta|^{-m}, 1\}$. *One give the generating functions of* (r,s)*-Horadam-Leonardo and* (r,s)*-Horadam-Leonardo-Lucas polynomials, respectively, as follows:*

(a)

$$\sum_{n=0}^{\infty} G_{mn+j} z^n = \frac{z^2\Theta_4+z\Theta_5+\Theta_6}{z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4}$$

where (as Theorem 8.1 (a))

$z^2\Theta_4 = z^2(G_{m+2}^2 G_{j+1} + (r+1)G_{m+1}^2 G_{j+2} + (-s)G_{m+1}^2 G_j + (-s)G_m^2 G_{j+2} - G_{m+1}G_{m+2}G_{j+2} + (s-r)G_m G_{m+1} G_{j+2} - (r+1)G_{m+1}G_{m+2}G_{j+1} - (s-r)G_m G_{m+2} G_{j+1} - (-s)G_m G_{m+1} G_{j+1} - (-s)G_m G_{m+2} G_j)$,

$z\Theta_5 = z(G_{m+2}G_{j+2} - (r+1)G_{m+1}G_{j+2} - (r+1)G_{m+2}G_{j+1} - (s-r)G_m G_{j+2} + (r+1)^2 G_{m+1}G_{j+1} - 2(-s)G_j G_{m+1} + ((-s)+(r+1)(s-r))G_m G_{j+1} + (r+1)(-s)G_m G_j)$,

$\Theta_6 = (-s)G_j$,

and

$z^3\Gamma_1 = z^3(-(-s)^{m+1})$,

$z^2\Gamma_2 = z^2((r+1)G_{m+2}^2 + ((r+1)^3+2(r+1)(s-r)+3(-s))G_{m+1}^2 + (-s)((r+1)^2+2(s-r))G_m^2 - 2((r+1)^2+(s-r))G_{m+2}G_{m+1} - (3(-s)+(r+1)(s-r))G_{m+2}G_m + ((r+1)^2(s-r)+2(s-r)^2-(r+1)(-s))G_{m+1}G_m)$,

$z\Gamma_3 = z((s-r)G_{m+2} - ((r+1)(s-r)+3(-s))G_{m+1} - ((s-r)^2 - 2(r+1)(-s))G_m)$,

$\Gamma_4 = (-s)$.

(b)

$$\sum_{n=0}^{\infty} H_{mn+j} z^n = \frac{z^2\Theta_4+z\Theta_5+\Theta_6}{z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4}$$

where (as Theorem 8.1 (a))

$z^2\Theta_4 = z^2((3H_{j+2} - 2(r+1)H_{j+1} - (s-r)H_j)H_{m+2}^2 + (((r+1)^2-(s-r))H_{j+2} + 3((r+1)(s-r)+(-s))H_{j+1} + ((s-r)^2+(r+1)(-s))H_j)H_{m+1}^2 + (-s)((r+1)H_{j+2} + 2(s-r)H_{j+1} + 3(-s)H_j)H_m^2 + (-4(r+1)H_{j+2} + 2((r+1)^2-(s-r))H_{j+1} + ((r+1)(s-r)-3(-s))H_j)H_{m+2}H_{m+1} + (-2(s-r)H_{j+2} + ((r+1)(s-r)-3(-s))H_{j+1} + 2(r+1)(-s)H_j)H_{m+2}H_m + (((r+1)(s-r)-3(-s))H_{j+2} + 2((s-r)^2+(r+1)(-s))H_{j+1} + 4(s-r)(-s)H_j)H_{m+1}H_m)$,

$z\Theta_5 = z((-2((r+1)^2+3(s-r))H_{j+2} + (2(r+1)^3+7(r+1)(s-r)+9(-s))H_{j+1} + ((r+1)^2(s-r)+4(s-r)^2-3(r+1)(-s))H_j)H_{m+2} + +((2(r+1)^3+7(r+1)(s-r)+9(-s))H_{j+2} - (2(r+1)^4+8(r+1)^2(s-r)+3(r+1)(-s)+2(s-r)^2)H_{j+1} - ((r+1)^3(s-r)+4(r+1)(s-r)^2-(r+1)^2(-s)+6(s-r)(-s))H_j)H_{m+1} + (((r+1)^2(s-r)-3(r+1)(-s)+4(s-r)^2)H_{j+2} - ((r+1)^3(s-r)+4(r+1)(s-r)^2-(r+1)^2(-s)+6(s-r)(-s))H_{j+1} - 2(-s)((r+1)^3+4(r+1)(s-r)+9(-s))H_j)H_m - 9(r+1)(-s)H_{j+1}H_{m+1})$,

$\Theta_6 = (4(r+1)^3(-s) - (r+1)^2(s-r)^2 - 4(s-r)^3 + 27(-s)^2 + 18(r+1)(s-r)(-s))H_j$,

and

$z^3\Gamma_1 = z^3(-(-s)^m(4(r+1)^3(-s) - (r+1)^2(s-r)^2 + 18(r+1)(s-r)(-s) - 4(s-r)^3 + 27(-s)^2))$,

$z^2\Gamma_2 = z^2((((r+1)^2+3(s-r))H_{m+2}^2 + ((r+1)^4+4(r+1)^2(s-r)+(s-r)^2+6(r+1)(-s))H_{m+1}^2 + (-s)((r+1)^3+4(r+1)(s-r)+9(-s))H_m^2 - (2(r+1)^3+7(r+1)(s-r)+9(-s))H_{m+2}H_{m+1} - ((r+1)^2(s-r)-3(r+1)(-s)+4(s-r)^2)H_{m+2}H_m + ((r+1)^3(s-r)+4(r+1)(s-r)^2-(r+1)^2(-s)+6(s-r)(-s))H_{m+1}H_m)$,

$z\Gamma_3 = z(-4(r+1)^3(-s) + (r+1)^2(s-r)^2 + 4(s-r)^3 - 27(-s)^2 - 18(r+1)(s-r)(-s))H_m$,

$\Gamma_4 = 4(r+1)^3(-s) - (r+1)^2(s-r)^2 - 4(s-r)^3 + 27(-s)^2 + 18(r+1)(s-r)(-s)$.

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Corollary 68.]. □

Some special cases of the last two Corollaries can be given as follows.

Corollary 9.4. *One gives the ordinary generating functions of the sequences* G_n, G_{2n}, G_{2n+1}, G_{-n}, G_{-2n}, G_{-2n+1} *and* H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} *as follows:*

(a) $(m=1, j=0, |z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, 1\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_n z^n &= \frac{z}{1-(r+1)z-(s-r)z^2+sz^3},\\ \sum_{n=0}^{\infty} H_n z^n &= \frac{3-2(r+1)z-(s-r)z^2}{1-(r+1)z-(s-r)z^2+sz^3}.\end{aligned}$$

(b) $(m=2, j=0, |z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, 1\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{2n} z^n &= \frac{(r+1)z+(-s)z^2}{1-(r^2+2s+1)z+(r^2+s^2+2s)z^2-s^2z^3},\\ \sum_{n=0}^{\infty} H_{2n} z^n &= \frac{3-2\left(r^2+2s+1\right)z+(r^2+s^2+2s)z^2}{1-(r^2+2s+1)z+(r^2+s^2+2s)z^2-s^2z^3}.\end{aligned}$$

(c) $(m=2, j=1, |z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, 1\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{2n+1} z^n &= \frac{1-(s-r)z}{1-(r^2+2s+1)z+(r^2+s^2+2s)z^2-s^2z^3},\\ \sum_{n=0}^{\infty} H_{2n+1} z^n &= \frac{(r+1)+(rs-2s-r-r^2)z-s(r-s)z^2}{1-(r^2+2s+1)z+(r^2+s^2+2s)z^2-s^2z^3}.\end{aligned}$$

(d) $(m=-1, j=0, |z| < \min\{|\alpha|, |\beta|, 1\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{-n} z^n &= \frac{z^2}{(-s)+(s-r)z+(r+1)z^2-z^3},\\ \sum_{n=0}^{\infty} H_{-n} z^n &= \frac{3(-s)+2(s-r)z+(r+1)z^2}{(-s)+(s-r)z+(r+1)z^2-z^3}.\end{aligned}$$

(e) $(m=-2, j=0, |z| < \min\{|\alpha|^2, |\beta|^2, 1\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{-2n} z^n &= \frac{(-s)z+(r+1)z^2}{s^2-(r^2+s^2+2s)z+(r^2+2s+1)z^2-z^3},\\ \sum_{n=0}^{\infty} H_{-2n} z^n &= \frac{3s^2-2(r^2+s^2+2s)z+(r^2+2s+1)z^2}{s^2-(r^2+s^2+2s)z+(r^2+2s+1)z^2-z^3}.\end{aligned}$$

(f) $(m=-2, j=1, |z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{-2n+1} z^n &= \frac{s^2-(r^2+s^2+2s)z+(r^2+r+s+1)z^2}{s^2-(r^2+s^2+2s)z+(r^2+2s+1)z^2-z^3},\\ \sum_{n=0}^{\infty} H_{-2n+1} z^n &= \frac{(r+1)(-s)^2-(r^3+r^2+3rs+rs^2+2s)z+(r^3+3rs+1)z^2}{s^2-(r^2+s^2+2s)z+(r^2+2s+1)z^2-z^3}.\end{aligned}$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Corollary 69.]. □

9.2 Generating Function of Generalized Horadam-Leonardo Polynomials(via Generalized Horadam-Leonardo Polynomials and (r,s)-Horadam-Leonardo Polynomials)

(In terms of elements of the sequence of generalized Horadam-Leonardo polynomials and (r,s)-Horadam-Leonardo polynomials) The ordinary generating function $\sum_{n=0}^{\infty} W_{mn+j}z^n$ of the sequence W_{mn+j} can be given as follows.

Lemma 9.5. *Suppose that* $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. *Assume that* $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j}z^n$ *is the generating function of the generalized Horadam-Leonardo polynomials* $\{W_{mn+j}\}$. *Then,* $\sum_{n=0}^{\infty} W_{mn+j}z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_{mn+j}z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 8.3 (a))

$z^2\Theta_4 = z^2(G_{m+2}^2W_{j+1} + ((r+1)W_{j+2} + (-s)W_j)G_{m+1}^2 + (-s)G_m^2W_{j+2} - ((r+1)W_{j+1} + W_{j+2})G_{m+1}G_{m+2} - ((s-r)W_{j+1} + (-s)W_j)G_mG_{m+2} + ((s-r)W_{j+2} - (-s)W_{j+1})G_mG_{m+1})$,

$z\Theta_5 = z((W_{j+2} - (r+1)W_{j+1})G_{m+2} + (-(r+1)W_{j+2} + (r+1)^2W_{j+1} - 2(-s)W_j)G_{m+1} + (-(s-r)W_{j+2} + ((-s)W_{j+1} + (r+1)(s-r)W_{j+1}) + (r+1)(-s)W_j)G_m)$,

$\Theta_6 = (-s)W_j$,

and

$z^3\Gamma_1 = z^3(-(-s)^{m+1})$,

$z^2\Gamma_2 = z^2((r+1)G_{m+2}^2 + ((r+1)^3 + 2(r+1)(s-r) + 3(-s))G_{m+1}^2 + ((r+1)^2(-s) + 2(s-r)(-s))G_m^2 - 2((r+1)^2 + (s-r))G_{m+1}G_{m+2} - ((r+1)(s-r) + 3(-s))G_mG_{m+2} + ((r+1)^2(s-r) + 2(s-r)^2 - (r+1)(-s))G_mG_{m+1})$,

$z\Gamma_3 = z((s-r)G_{m+2} - ((r+1)(s-r) + 3(-s))G_{m+1} + (2(r+1)(-s) - (s-r)^2)G_m)$,

$\Gamma_4 = (-s)$.

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Lemma 70]. □

As particular example (generating functions of (r,s)-Horadam-Leonardo-Lucas polynomials), Lemma 9.5 gives the following result

Corollary 9.6. *Suppose that* $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. *one can gives the generating function of* (r,s)-*Horadam-Leonardo-Lucas polynomials as follows:*

$$\sum_{n=0}^{\infty} H_{mn+j}z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 8.3 (a))

$z^2\Theta_4 = z^2(G_{m+2}^2H_{j+1} + ((r+1)H_{j+2} + (-s)H_j)G_{m+1}^2 + (-s)G_m^2H_{j+2} - ((r+1)H_{j+1} + H_{j+2})G_{m+1}G_{m+2} - ((s-r)H_{j+1} + (-s)H_j)G_mG_{m+2} + ((s-r)H_{j+2} - (-s)H_{j+1})G_mG_{m+1})$,

$z\Theta_5 = z((H_{j+2} - (r+1)H_{j+1})G_{m+2} + (-(r+1)H_{j+2} + (r+1)^2H_{j+1} - 2(-s)H_j)G_{m+1} + (-(s-r)H_{j+2} + ((-s)H_{j+1} + (r+1)(s-r)H_{j+1}) + (r+1)(-s)H_j)G_m)$,

$\Theta_6 = (-s)H_j$,

and

$z^3\Gamma_1 = z^3(-(-s)^{m+1})$,

$z^2\Gamma_2 = z^2((r+1)G_{m+2}^2 + ((r+1)^3 + 2(r+1)(s-r) + 3(-s))G_{m+1}^2 + ((r+1)^2(-s) + 2(s-r)(-s))G_m^2 - 2((r+1)^2 + (s-r))G_{m+1}G_{m+2} - ((r+1)(s-r) + 3(-s))G_mG_{m+2} + ((r+1)^2(s-r) + 2(s-r)^2 - (r+1)(-s))G_mG_{m+1})$,

$z\Gamma_3 = z((s-r)G_{m+2} - ((r+1)(s-r) + 3(-s))G_{m+1} + (2(r+1)(-s) - (s-r)^2)G_m)$,

$\Gamma_4 = (-s)$.

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Corollary 71.]. □

9.3 Generating Function of Generalized Horadam-Leonardo Polynomials(via Generalized Horadam-Leonardo Polynomials and (r,s)-Horadam-Leonardo-Lucas Polynomials)

(In terms of elements of the sequence of generalized Horadam-Leonardo polynomials and (r,s)-Horadam-Leonardo-Lucas polynomials) The ordinary generating function $\sum_{n=0}^{\infty} W_{mn+j}z^n$ of the sequence W_{mn+j} can be given as follows.

Lemma 9.7. *Let $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. Assume that $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j}z^n$ is the generating function of the generalized Horadam-Leonardo polynomials $\{W_{mn+j}\}$. Then, $\sum_{n=0}^{\infty} W_{mn+j}z^n$ is given by*

$$\sum_{n=0}^{\infty} W_{mn+j}z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 8.5 (a))

$z^2\Theta_4 = z^2((3W_{j+2}-2(r+1)W_{j+1}-(s-r)W_j)H_{m+2}^2+(((r+1)^2-(s-r))W_{j+2}+3((r+1)(s-r)+(-s))W_{j+1}+((s-r)^2+(r+1)(-s))W_j)H_{m+1}^2+(-s)((r+1)W_{j+2}+2(s-r)W_{j+1}+3(-s)W_j)H_m^2+(-4(r+1)W_{j+2}+2((r+1)^2-(s-r))W_{j+1}+((r+1)(s-r)-3(-s))W_j)H_{m+2}H_{m+1}+(-2(s-r)W_{j+2}+((r+1)(s-r)-3(-s))W_{j+1}+2(r+1)(-s)W_j)H_{m+2}H_m+(((r+1)(s-r)-3(-s))W_{j+2}+2((s-r)^2+(r+1)(-s))W_{j+1}+4(s-r)(-s)W_j)H_{m+1}H_m)$,

$z\Theta_5 = z((-2((r+1)^2+3(s-r))W_{j+2}+(2(r+1)^3+7(r+1)(s-r)+9(-s))W_{j+1}+((r+1)^2(s-r)-3(r+1)(-s)+4(s-r)^2)W_j)H_{m+2}+((2(r+1)^3+7(r+1)(s-r)+9(-s))W_{j+2}-2((r+1)^4+4(r+1)^2(s-r)+6(r+1)(-s)+(s-r)^2)W_{j+1}-((r+1)^3(s-r)+4(r+1)(s-r)^2-(r+1)^2(-s)+6(s-r)(-s))W_j)H_{m+1}+(((r+1)^2(s-r)+4(s-r)^2-3(r+1)(-s))W_{j+2}-(4(r+1)(s-r)^2+(r+1)^3(s-r)-(r+1)^2(-s)+6(s-r)(-s))W_{j+1}-2(-s)((r+1)^3+4(r+1)(s-r)+9(-s))W_j)H_m)$,

$\Theta_6 = (4(r+1)^3(-s)-(r+1)^2(s-r)^2-4(s-r)^3+27(-s)^2+18(r+1)(s-r)(-s))W_j$,

and

$z^3\Gamma_1 = z^3(-(-s)^m(4(r+1)^3(-s)-(r+1)^2(s-r)^2+18(r+1)(s-r)(-s)-4(s-r)^3+27(-s)^2))$,

$z^2\Gamma_2 = z^2(((r+1)^2+3(s-r))H_{m+2}^2+((r+1)^4+4(r+1)^2(s-r)+(s-r)^2+6(r+1)(-s))H_{m+1}^2+(-s)((r+1)^3+4(r+1)(s-r)+9(-s))H_m^2-(2(r+1)^3+7(r+1)(s-r)+9(-s))H_{m+2}H_{m+1}-((r+1)^2(s-r)+4(s-r)^2-3(r+1)(-s))H_{m+2}H_m+(r+1)(s-r)((r+1)^2+4(s-r))H_{m+1}H_m-(-s)((r+1)^2-6(s-r))H_{m+1}H_m)$,

$z\Gamma_3 = z(-4(r+1)^3(-s)+(r+1)^2(s-r)^2+4(s-r)^3-27(-s)^2-18(r+1)(s-r)(-s))H_m$,

$\Gamma_4 = 4(r+1)^3(-s)-(r+1)^2(s-r)^2+18(r+1)(s-r)(-s)-4(s-r)^3+27(-s)^2$.

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Lemma 72.]. □

As particular example (generating function of (r,s)-Horadam-Leonardo polynomials), from Lemma 9.7, one gets the following result.

Corollary 9.8. *Suppose that $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. One gives generating function of (r,s)-Horadam-Leonardo polynomials as follows:*

$$\sum_{n=0}^{\infty} G_{mn+j}z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 8.5 (a))

$z^2\Theta_4 = z^2((3G_{j+2}-2(r+1)G_{j+1}-(s-r)G_j)H_{m+2}^2+(((r+1)^2-(s-r))G_{j+2}+3((r+1)(s-r)+(-s))G_{j+1}+((s-r)^2+(r+1)(-s))G_j)H_{m+1}^2+(-s)((r+1)G_{j+2}+2(s-r)G_{j+1}+3(-s)G_j)H_m^2+(-4(r+1)G_{j+2}+2((r+1)^2-(s-r))G_{j+1}+((r+1)(s-r)-3(-s))G_j)H_{m+2}H_{m+1}+(-2(s-r)G_{j+2}+((r+1)(s-r)-3(-s))G_{j+1}+2(r+1)(-s)G_j)H_{m+2}H_m+(((r+1)(s-r)-3(-s))G_{j+2}+2((s-r)^2+(r+1)(-s))G_{j+1}+4(s-r)(-s)G_j)H_{m+1}H_m)$,

$z\Theta_5 = z((-2((r+1)^2+3(s-r))G_{j+2}+(2(r+1)^3+7(r+1)(s-r)+9(-s))G_{j+1}+((r+1)^2(s-r)-3(r+1)(-s)+4(s-r)^2)G_j)H_{m+2}+((2(r+1)^3+7(r+1)(s-r)+9(-s))G_{j+2}-2((r+1)^4+4(r+1)^2(s-r)+6(r+1)(-s)+(s-r)^2)G_{j+1}-((r+1)^3(s-r)+4(r+1)(s-r)^2-(r+1)^2(-s)+6(s-r)(-s))G_j)H_{m+1}+(((r+1)^2(s-r)+4(s-r)^2-3(r+1)(-s))G_{j+2}-(4(r+1)(s-r)^2+(r+1)^3(s-r)-(r+1)^2(-s)+6(s-r)(-s))G_{j+1}-2(-s)((r+1)^3+4(r+1)(s-r)+9(-s))G_j)H_m),$

$\Theta_6 = (4(r+1)^3(-s)-(r+1)^2(s-r)^2-4(s-r)^3+27(-s)^2+18(r+1)(s-r)(-s))G_j,$

and

$z^3\Gamma_1 = z^3(-(-s)^m(4(r+1)^3(-s)-(r+1)^2(s-r)^2+18(r+1)(s-r)(-s)-4(s-r)^3+27(-s)^2)),$

$z^2\Gamma_2 = z^2(((r+1)^2+3(s-r))H_{m+2}^2+((r+1)^4+4(r+1)^2(s-r)+(s-r)^2+6(r+1)(-s))H_{m+1}^2+(-s)((r+1)^3+4(r+1)(s-r)+9(-s))H_m^2-(2(r+1)^3+7(r+1)(s-r)+9(-s))H_{m+2}H_{m+1}-((r+1)^2(s-r)+4(s-r)^2-3(r+1)(-s))H_{m+2}H_m+(r+1)(s-r)((r+1)^2+4(s-r))H_{m+1}H_m-(-s)((r+1)^2-6(s-r))H_{m+1}H_m),$

$z\Gamma_3 = z(-4(r+1)^3(-s)+(r+1)^2(s-r)^2+4(s-r)^3-27(-s)^2-18(r+1)(s-r)(-s))H_m,$

$\Gamma_4 = 4(r+1)^3(-s)-(r+1)^2(s-r)^2+18(r+1)(s-r)(-s)-4(s-r)^3+27(-s)^2.$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [25, Corollary 73.]. □

10 CONCLUSIONS

The Fibonacci and Lucas sequences are sources of many nice and interesting identities. For the applications of these second order sequences in science and nature, one refer the citations in [11,12,10].

In this study, we define and investigate a linear third order polynomial (and two special cases of its). We called them the generalized Horadam-Leonardo polynomials (and (r, s)-Horadam-Leonardo and (r, s)-Horadam-Leonardo-Lucas polynomials). Binet's formulas, generating functions, Simson formulas, and the summation formulas for these polynomial sequences are presented. Then, some identities and matrices related to these polynomials are given.

Linear recurrence relations (sequences) have many applications. Next, we list applications of sequences which are linear recurrence relations.

First, some applications of second order sequences are given. For the applications of

- Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [26].
- Pell Numbers to the solutions of three-dimensional difference equation systems, see [27].
- Jacobsthal numbers to special matrices, see [28].
- generalized k-order Fibonacci numbers to hybrid quaternions, see [29].
- Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [30].
- generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [31].
- generalized Fibonacci numbers to binomial sums, see [32].
- generalized Jacobsthal numbers to hyperbolic numbers, see [33].
- generalized Fibonacci numbers to dual hyperbolic numbers, see [34].
- Laplace transform and various matrix operations to the characteristic polynomial of the Fibonacci numbers, see [35].
- generalized Fibonacci Matrices to Cryptography, see [36].
- higher order Jacobsthal numbers to quaternions, see [37].
- Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [38].
- Fibonacci numbers to lacunary statistical convergence, see [39].
- Fibonacci numbers to lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [40].
- Fibonacci numbers to ideal convergence on intuitionistic fuzzy normed linear spaces, see [41].
- k-Fibonacci and $k-$Lucas numbers to spinors, see [42].
- dual-generalized complex Fibonacci and Lucas numbers to Quaternions, see [43].
- Hyperbolic Fibonacci numbers to Quaternions, see [44].

Now, some applications of third order sequences are given. For the applications of

- third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [45] and [46], respectively.
- Tribonacci numbers to special matrices, see [47].
- Padovan numbers and Tribonacci numbers to coding theory, see [48] and [49], respectively.
- Pell-Padovan numbers to groups, see [50].
- adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [51].
- Gaussian Tribonacci numbers to various graphs, see [52].
- third-order Jacobsthal numbers to hyperbolic numbers, see [53].
- Narayan numbers to finite groups see [54].
- generalized third-order Jacobsthal sequence to binomial transform, see [55].
- generalized Generalized Padovan numbers to Binomial Transform, see [56].
- generalized Tribonacci numbers to Gaussian numbers, see [57].
- generalized Tribonacci numbers to Sedenions, see [58].
- Tribonacci and Tribonacci-Lucas numbers to matrices, see [59].
- generalized Tribonacci numbers to circulant matrix, see [60].
- Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see [61].
- hyperbolic Leonardo and hyperbolic Francois numbers to quaternions, see [62].

Next, list of some applications of fourth order sequences are given. For the applications of

- Tetranacci and Tetranacci-Lucas numbers to quaternions, see [63].
- generalized Tetranacci numbers to Gaussian numbers, see [64].
- Tetranacci and Tetranacci-Lucas numbers to matrices, see [65].
- generalized Tetranacci numbers to binomial transform, see [66].

Now, some applications of fifth order sequences are given. For the applications of

- Pentanacci numbers to matrices, see [67].
- generalized Pentanacci numbers to quaternions, see [68].
- generalized Pentanacci numbers to binomial transform, see [69].

  Next, some applications of second order sequences of polynomials are given. For the applications of

- generalized Fibonacci Polynomials to the summation formulas, see [70].
- generalized Fibonacci Polynomials, see [4].

  Now, some applications of third order sequences of polynomials are given. For the applications of

- generalized Tribonacci Polynomials, see [25].

COMPETING INTERESTS

Author has declared that no competing interests exist.

REFERENCES

[1] Frei G. Binary lucas and fibonacci polynomials I. Math. Nadir. 1980;96:83-112.

[2] Flórez R, McAnally N, Mukherjee A. Identities for the generalized fibonacci polynomial. Integers. 2018;18B.

[3] He TX, Peter JS, Shiue PJS. On sequences of numbers and polynomials defined by linear recurrence relations of order 2. International Journal of Mathematics and Mathematical Sciences. 2009;21. Article ID 709386 DOI:10.1155/2009/709386

[4] Soykan Y. On generalized fibonacci polynomials: Horadam polynomials. Earthline Journal of Mathematical Sciences. 2023;11(1):23-114. E-ISSN: 2581-8147 DOI:https://doi.org/10.34198/ejms.11123.23114

[5] Soykan Y. Sums and generating functions of generalized fibonacci polynomials via matrix methods. International Journal of Advances in Applied Mathematics and Mechanics. 2023;10(4):23-71.

[6] Wang J. Some new results for the (p,q)-fibonacci and lucas polynomials. Advances in Difference Equations; 2014. Available: http://www.advancesindifferenceequations.com/content/2014

[7] Wang W, Wang H. Generalized-humbert polynomials via generalized fibonacci polynomials. Applied Mathematics and Computation. 2017;307:204-216.
Available: https://doi.org/10.1016/j.amc.2017.02.050

[8] Andrica D, Bagdasar O. Recurrent sequences key results, applications, and problems. Springer; 2020.

[9] Djordjević GB, Milovanović GV. Special classes of polynomials. University of Niš, Faculty of Technology, Leskovac; 2014.
Available: http://www.mi.sanu.ac.rs/˜gvm/Teze/Special%20Classes%20of

[10] Koshy T. Pell and pell-lucas numbers with applications. Springer, New York; 2014.

[11] Koshy T. Fibonacci and lucas numbers with applications. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Second Edition, John Wiley&Sons, New York. 2018;1.

[12] Koshy T. Fibonacci and lucas numbers with applications. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts, John Wiley&Sons, New York. 2019;2.

[13] Soykan Y. Generalized fibonacci numbers: Sum formulas. Minel Yayın; 2022.
Available: https://www.minelyayin.com/generalized-fibonacci-numbers-sum-formulas-51

[14] Vajda S. Fibonacci and lucas numbers, and the golden section: Theory and applications. Dover Publications Inc.; 2008.

[15] Cerda-Moralez G. On third-order jacobsthal polynomials and their properties. Miskolc Mathematical Notes. 2021;22(1):123-132.
DOI: 10.18514/MMN.2021.3227

[16] Merzouka H, Boussayoudb A, Chelghamc M. Generating functions of generalized tribonacci and tricobsthal polynomials. Montes Taurus Journal of Pure and Applied Mathematics. 2020;2(2):7-37.

[17] Özkan E, Altun İ. Generalized lucas polynomials and relationships between the fibonacci polynomials and lucas polynomials. Communications in Algebra. 2019;47(10).
DOI: 10.1080/00927872.2019.1576186

[18] Ricci PE. A note on q-matrices and higher order fibonacci polynomials. Notes on Number Theory and Discrete Mathematics. 2021;27(1):91-100.
DOI: 10.7546/nntdm.2021.27.1.91-100

[19] Soykan, Y. generalized leonardo numbers. Journal of Progressive Research in Mathematics. 2021;18(4):58-84.

[20] Soykan Y. Generalized john numbers. Journal of Progressive Research in Mathematics. 2022;19(1):17-34.

[21] Soykan Y. Generalized ernst Numbers. Asian Journal of Pure and Applied Mathematics. 2022;4(3):1-15.

[22] Soykan Y, Okumuş İ, Taşdemir E. Generalized pisano numbers. Notes on Number Theory and Discrete Mathematics. 2022;28(3):477-490.
DOI: 10.7546/nntdm.2022.28.3.477-490

[23] Soykan Y. Generalized edouard Numbers. International Journal of Advances in Applied Mathematics and Mechanics. 2022;9(3):41-52.

[24] Soykan Y, Okumuş İ, Bilgin NG. On generalized bigollo numbers. Asian Research Journal of Mathematics. 2023;19(8):72-88.
DOI: 10.9734/ARJOM/2023/v19i8690

[25] Soykan Y. Generalized tribonacci polynomials. Earthline Journal of Mathematical Sciences. 2023;13(1):1-120.
DOI: https://doi.org/10.34198/ejms.13123.1120

[26] Azak AZ. Pauli gaussian fibonacci and pauli gaussian lucas quaternions. Mathematics. 2022;10:4655.
DOI: https://doi.org/10.3390/math10244655

[27] Büyük H, Taşkara N. On the solutions of three-dimensional difference equation systems via pell numbers. European Journal of Science and Technology. 2022;34(Special Issue):433-440.

[28] Vasanthi S, Sivakumar B. Jacobsthal matrices and their properties. Indian Journal of Science and Technology. 2022;15(5):207-215.
DOI:https://doi.org/10.17485/IJST/v15i5.1948

[29] Gül K. Generalized k-order fibonacci hybrid quaternions. Erzincan University Journal of Science and Technology. 2022;15(2):670-683.
DOI: 10.18185/erzifbed.1132164

[30] Yılmaz N. Split complex bi-periodic fibonacci and lucas numbers. Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 2022;71(1):153-164.
DOI:10.31801/cfsuasmas.704435

[31] Yılmaz N. The generalized bivariate fibonacci and lucas matrix polynomials. Mathematica Montisnigri. 2022;LIII:33-44.
DOI: 10.20948/mathmontis-2022-53-5

[32] Ulutaş YT, Toy D. Some equalities and binomial sums about the generalized fibonacci number u_n. Notes on Number Theory and Discrete Mathematics. 2022;28(2):252-260. DOI: 10.7546/nntdm.2022.28.2.252-260

[33] Soykan Y, Taşdemir E. A study on hyperbolic numbers with generalized jacobsthal numbers components. International Journal of Nonlinear Analysis and Applications. 2022;13(2):1965-1981. DOI: http://dx.doi.org/10.22075/ijnaa.2021.22113.2328

[34] Soykan Y. On dual hyperbolic generalized fibonacci numbers. Indian J Pure Appl Math; 2021. DOI: https://doi.org/10.1007/s13226-021-00128-2

[35] Deveci Ö, Shannon AG. On recurrence results from matrix transforms. Notes on Number Theory and Discrete Mathematics. 2022;28(4):589-592. DOI: 10.7546/nntdm.2022.28.4.589-592

[36] Prasad K, Mahato H. Cryptography using generalized fibonacci matrices with affine-hill cipher. Journal of Discrete Mathematical Sciences & Cryptography. 2022;25(8-A): 2341-2352. DOI : 10.1080/09720529.2020.1838744

[37] Özkan E, Uysal M. On quaternions with higher order jacobsthal numbers components. Gazi University Journal of Science. 2023;36(1):336-347. DOI: 10.35378/gujs. 1002454

[38] Goy T, Shattuck M. Fibonacci and lucas identities from toeplitz-hessenberg matrices. Appl. Appl. Math. 2019;14(2):699-715.

[39] Bilgin NG. Fibonacci lacunary statistical convergence of order γ in IFNLS. International Journal of Advances in Applied Mathematics and Mechanics. 2021;8(4):28-36.

[40] Kişi Ö, Tuzcuoglu I. Fibonacci lacunary statistical convergence in intuitionistic fuzzy normed linear spaces. Journal of Progressive Research in Mathematics. 2020;16(3):3001-3007.

[41] Kişi Ö, Debnathb P. Fibonacci ideal convergence on intuitionistic fuzzy normed linear spaces. Fuzzy Information and Engineering. 2022;1-13. DOI: https://doi.org/10.1080/16168658.2022.2160226

[42] Kumari M, Prasad K, Frontczak R. On the k-fibonacci and $k-$lucas spinors. Notes on Number Theory and Discrete Mathematics. 2023;29(2):322-335. DOI: 10.7546/nntdm.2023.29.2.322-335

[43] Şentürk GY, Gürses N, Yüce S. Construction of dual-generalized complex fibonacci and lucas quaternions. Carpathian Math. Publ. 2022;14(2):406-418. DOI:10.15330/cmp.14.2.406-418

[44] Daşdemir A. On recursive hyperbolic fibonacci quaternions. Communications in Advanced Mathematical Sciences. 2021;4(4):198-207. DOI:10.33434/cams.997824%20Polynomials.pdf

[45] Cerda-Morales G. Identities for third order jacobsthal quaternions. Advances in Applied Clifford Algebras. 2017;27(2):1043-1053.

[46] Cerda-Morales G. On a generalization of tribonacci quaternions. Mediterranean Journal of Mathematics. 2017;14(239):1-12.

[47] Yilmaz N, Taskara N. Tribonacci and tribonacci-lucas numbers via the determinants of special matrices. Applied Mathematical Sciences. 2014;8(39):1947-1955.

[48] Shtayat J, Al-Kateeb A. An encoding-decoding algorithm based on Padovan numbers; 2019. arXiv:1907.02007

[49] Basu M, Das M. Tribonacci matrices and a new coding theory. Discrete Mathematics, Algorithms and Applications. 2014;6(1):1450008(17 pages).

[50] Deveci Ö, Shannon AG. Pell–Padovan-Circulant sequences and their applications. Notes on Number Theory and Discrete Mathematics. 2017;23(3):100-114.

[51] Göcen M. The exact solutions of some difference equations associated with adjusted Jacobsthal-Padovan numbers. Kırklareli University Journal of Engineering and Science. 2022;8(1):1-14. DOI: 10.34186/klujes.1078836

[52] Sunitha K, Sheriba M. Gaussian tribonacci r-graceful labeling of some tree related graphs. Ratio Mathematica. 2022;44:188-196.

[53] Dikmen CM, Altınsoy M. On third order hyperbolic jacobsthal numbers. Konuralp Journal of Mathematics. 2022;10(1):118-126.

[54] Kuloğlu B, Özkan E, Shannon AG. The narayana sequence in finite groups. Fibonacci Quarterly. 2022;60(5):212-221.

[55] Soykan Y, Taşdemir E, Göcen M. Binomial transform of the generalized third-order jacobsthal sequence. Asian-European Journal of Mathematics. 2022;15(12). DOI: https://doi.org/10.1142/S1793557122502242

[56] Soykan Y, Taşdemir E, Okumuş İ. A study on binomial transform of the generalized Padovan Sequence. Journal of Science and Arts. 2022;22(1):63-90.
DOI: https://doi.org/10.46939/J.Sci.Arts-22.1-a06

[57] Soykan Y, Taşdemir E, Okumuş İ, Göcen M. Gaussian generalized tribonacci numbers. Journal of Progressive Research in Mathematics (JPRM). 2018;14(2):2373-2387.

[58] Soykan Y, Okumuş İ, Taşdemir E. On generalized tribonacci sedenions. Sarajevo Journal of Mathematics. 2020;16(1):103-122. ISSN 2233-1964
DOI: 10.5644/SJM.16.01.08

[59] Soykan Y. Matrix sequences of tribonacci and tribonacci-lucas numbers. Communications in Mathematics and Applications. 2020;11(2):281-295.
DOI: 10.26713/cma.v11i2.1102

[60] Soykan Y. Explicit euclidean norm, eigenvalues, spectral norm and determinant of circulant matrix with the generalized tribonacci numbers. Earthline Journal of Mathematical Sciences. 2021;6(1):131-151.
DOI: https://doi.org/10.34198/ejms.6121.131151

[61] Taşyurdu Y, Polat YE. Tribonacci and tribonacci-lucas hybrinomials. Journal of Mathematics Research. 2021;13(5).

[62] Dişkaya O, Menken H, Catarino PMMC. On the hyperbolic leonardo and hyperbolic francois quaternions. Journal of New Theory. 2023;42:74-85.

[63] Soykan Y. Tetranacci and tetranacci-lucas quaternions. Asian Research Journal of Mathematics. 2019;15(1):1-24. Article no.ARJOM.50749

[64] Soykan Y. Gaussian generalized tetranacci numbers. Journal of Advances in Mathematics and Computer Science. 2019;31(3):1-21. Article no.JAMCS.48063

[65] Soykan Y. Matrix sequences of tetranacci and tetranacci-lucas numbers. Int. J. Adv. Appl. Math. and Mech. 2019;7(2):57-69. ISSN: 2347-2529

[66] Soykan Y. On binomial transform of the generalized tetranacci sequence. International Journal of Advances in Applied Mathematics and Mechanics. 2021;9(2):8-27.

[67] Sivakumar B, James V. A notes on matrix sequence of pentanacci numbers and pentanacci cubes. Communications in Mathematics and Applications. 2022;13(2):603-611.
DOI: 10.26713/cma.v13i2.1725

[68] Soykan Y, Özmen N, Göcen M. On generalized pentanacci quaternions. Tbilisi Mathematical Journal. 2020;13(4):169-181.

[69] Soykan Y. Binomial transform of the generalized pentanacci sequence. Asian Research Journal of Current Science. 2021;3(1):209-231.

[70] Soykan Y. A study on generalized fibonacci polynomials: Sum formulas. International Journal of Advances in Applied Mathematics and Mechanics. 2022;10(1):39-118. ISSN: 2347-2529

Peer-review history:
The peer review history for this paper can be accessed here:
http://www.sdiarticle5.com/review-history/100417