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Strong Consistency of a Kernel-type Estimator for the Intensity Obtained as the Product of a Periodic Function with the Power Function Trend of a Non-homogeneous Poisson Process

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Authors' contributions

This work was carried out in collaboration among all authors. Author IWM designed the study and managed literature searches. Author IM performed the mathematical analysis and wrote the first draft of the manuscript with help of authors IWM and HS. All authors read and correct the draft and approved the final manuscript.

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ABSTRACT

In [1], a kernel-type estimator for the intensity obtained as the product of a periodic function with the power function trend of a non-homogeneous Poisson process has been formulated. In addition, asymptotic approximations to the bias, variance and mean squared error of this estimator have been established. In this paper, we construct a proof of strong consistency of the estimator proposed in [1].

Keywords: Poisson process; periodic intensity function; power function trend; strong consistency; complete convergence.

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1. INTRODUCTION AND MAIN RESULTS

Let N be a non-homogeneous Poisson process on $[0,\infty]$ having (unknown) locally integrable intensity function λ. We assume the intensity function to be a product of a periodic function with the power function trend. That is, the equation

$$\lambda(s) = \left(\lambda_c^*(s)\right)as^b, \tag{1.1}$$

holds true for each point $s \in [0,\infty]$, where $\lambda_c^*(s)$ is a periodic function with known period τ, as^b is the power function trend with $b > 0$ (known), and a denotes the slope of the power function trend. Without loss of generality, the intensity function given in (1.1) can also be written as

$$\lambda(s) = \left(\lambda_c\ (s)\right)s^b, \tag{1.2}$$

where $\lambda_c\ (s) = a\left(\lambda_c^*(s)\right)$ is also a periodic function with period τ. Hence, for each point $s \in [0,\infty)$ and for each integer k, we have

$$\lambda_c\ (s + k\tau) = \lambda_c\ (s). \tag{1.3}$$

In [1], a kernel-type estimator for the intensity obtained as the product of a periodic function with the power function trend of a non homogeneous Poisson process has been formulated. In addition, asymptotic approximations to the bias, variance, and mean squared error of this estimator have been established. Strong consistency of this estimator is still an open problem. In this paper, we construct a proof of strong consistency of the estimator proposed in [1]. The estimator has been formulated as follows,

$$\hat{\lambda}_{c,n,K}(s) = \frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n(s+k\tau)^b}\int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right)N(dx), \tag{1.4}$$

where τ is the period of λ_c , b is the power of the trend function, h_n is bandwidth which satisfies $\lim_{n\to\infty} h_n = 0$, and K is a kernel function which satisfies three conditions, (K1) K is a probability density function, (K2) K is bounded, and (K3) K has (closed) support $[-1,1]$.

Since strong consistency of an estimator is implied by complete convergence of that estimator, first we establish the complete convergence of the estimator given in (1.4), which is presented in the following theorem. Some related results can be found in [2-10].

Theorem 1.1 (Complete convergence)

Suppose that the intensity function λ satisfies (1.2) and is locally integrable. If the kernel K satisfies conditions (K1), (K2), (K3), the bandwidth $h_n = n^{-\alpha}$ where $0 < \alpha < 1$ and $\alpha < b$, then

$$\hat{\lambda}_{c,n,K}(s) \xrightarrow{c} \lambda_c(s)$$

as $n \to \infty$, provided s is a Lebesgue point of λ. In other words, $\hat{\lambda}_{c,n,K}(s)$ converges completely to λ_c as $n \to \infty$.

Corollary 1.2 (Strong consistency)

Suppose that the intensity function λ satisfies (1.2) and is locally integrable. If the kernel K satisfies conditions (K1), (K2), (K3), the bandwidth $h_n = n^{-\alpha}$ where $0 < \alpha < 1$ for $\alpha < b$, then

$$\hat{\lambda}_{c,n,K}(s) \xrightarrow{a.s.} \lambda_c(s)$$

as $n \to \infty$, provided s is a Lebesgue point of λ. In other words, $\hat{\lambda}_{c,n,K}(s)$ is a strong consistent estimator of $\lambda_c(s)$.

2. SOME TECHNICAL LEMMAS

This following Lemma is needed for proving Theorem 1.1.

Lemma 2.1 (Asymptotic unbiasedness)

Suppose that the intensity function λ satisfies (1.2) and is locally integrable, $h_n \to 0$, the kernel K satisfies (K1), (K2), and (K3), then

$$E\hat{\lambda}_{c,n,K}(s) \to \lambda_c(s)$$

as $n \to \infty$.

Proof. The expectation of $\hat{\lambda}_{c,n,K}(s)$ (cf. (1.4)) can be computed as follows

$$E\hat{\lambda}_{c,n,K}(s) = \frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n(s+k\tau)^b}\int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)I(x \in [0,n])dx. \tag{2.1}$$

Let $y = x - (s + k\tau)$, then the r.h.s of (2.1) can be written as

$$\frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n(s+k\tau)^b}\int_{\mathbb{R}} K\left(\frac{y}{h_n}\right)\lambda(y+s+k\tau)I(y+s+k\tau \in [0,n])dy, \tag{2.2}$$

Where I denotes the indicator function. Since λ satisfies (1.2) and (1.3), then the quantity in (2.2) becomes

$$\frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n(s+k\tau)^b}\int_{\mathbb{R}} K\left(\frac{y}{h_n}\right)\lambda_c(y+s)(y+s+k\tau)^b I(y+s+k\tau \in [0,n])dy$$

$$= \frac{\tau}{nh_n}\int_{\mathbb{R}} K\left(\frac{y}{h_n}\right)\lambda_c(y+s)\sum_{k=0}^{\infty}\frac{(y+s+k\tau)^b}{(s+k\tau)^b}I(y+s+k\tau \in [0,n])dy.$$

By a simple calculation, we obtain

$$\sum_{k=0}^{\infty}\frac{(y+s+k\tau)^b}{(s+k\tau)^b}I(y+s+k\tau \in [0,n]) = \frac{n}{\tau} + O(1),$$

as $n \to \infty$. Hence, we have

$$E\hat{\lambda}_{c,n,K}(s) = \frac{1}{h_n}\int_{\mathbb{R}} K\left(\frac{y}{h_n}\right)\lambda_c(y+s)\,dy + O\left(\frac{1}{n}\right) \tag{2.3}$$

as $n \to \infty$. The first term on the r.h.s of (2.3) can be written as

$$\frac{1}{h_n}\int_{\mathbb{R}} K\left(\frac{y}{h_n}\right)\big(\lambda_c(y+s) - \lambda_c(s) + \lambda_c(s)\big)\,dy = \frac{1}{h_n}\int_{\mathbb{R}} K\left(\frac{y}{h_n}\right)\big(\lambda_c(y+s) - \lambda_c(s)\big)\,dy + \frac{1}{h_n}\int_{\mathbb{R}} K\left(\frac{y}{h_n}\right)\lambda_c(s)\,dy. \tag{2.4}$$

Since K satisfies (K2) which implies there exists a real number M such that $K(x) \leq M$ and K satisfies (K3), and also s is Lebesgue point of λ which implies a Lebesgue point of λ_c as well, then the first term on the r.h.s of (2.4) does not exceed

$$\frac{1}{h_n}\int_{\mathbb{R}} M\big(\lambda_c(y+s) - \lambda_c(s)\big)\,dy$$

$$\leq 2\left(\frac{M}{2h_n}\right)\int_{\mathbb{R}} |\lambda_c(y+s) - \lambda_c(s)|\,dy = o(1) \tag{2.5}$$

as $n \to \infty$. Since K satisfies (K1), then the second term on the r.h.s of (2.4), can be written as

$$\frac{\lambda_c(s)}{h_n}\int_{\mathbb{R}} K\left(\frac{y}{h_n}\right)dy = \lambda_c(s). \tag{2.6}$$

From (2.5) and (2.6),we see that the first term on the r.h.s. of (2.3) is equal to $\lambda_c(s) + o(1)$, as $n \to \infty$. Clearly, the second term on the r.h.s. of (2.3) is $o(1)$, as $n \to \infty$. This completes the proof Lemma 2.1.

Lemma 2.2 (Asymptotic approximation to the variance)

Suppose that intensity function λ satisfies (1.2) and is locally integrable, the kernel function K satisfies (K1), (K2), and (K3), the bandwidth $h_n \to 0$, and s is a Lebesgue point of λ.

If $0 < b < 1$ and $n^{b+1}h_n \to \infty$, as $n \to \infty$, then

$$Var(\hat{\lambda}_{c,n,K}(s)) = \frac{\tau\lambda_c(s)}{n^{b+1}h_n(1-b)}\int_{-1}^{1} K^2(x)dx + o\left(\frac{1}{n^{b+1}h_n}\right), \tag{2.7}$$

as $n \to \infty$.

If $b = 1$ and $\frac{n^2h_n}{\ln(n)} \to \infty$, as $n \to \infty$,then

$$Var(\hat{\lambda}_{c,n,K}(s)) = \frac{\tau\lambda_c(s)ln(n)}{n^2h_n}\int_{-1}^{1} K^2(x)dx + o\left(\frac{ln(n)}{n^2h_n}\right), \tag{2.8}$$

as $n \to \infty$.

If $b > 1$ and $n^2h_n \to \infty$, as $n \to \infty$, then

$$Var\left(\hat{\lambda}_{c,n,K}(s)\right) = \frac{\tau^{2-b}\lambda_c(s)\zeta(b)}{n^2h_n}\int_{-1}^{1} K^2(x)dx + o\left(\frac{1}{n^2h_n}\right), \tag{2.9}$$

as $n \to \infty$, where $\zeta(b) = \lim_{n\to\infty}\Big(\sum_{k=1}^{\infty}\frac{1}{k^b}I(y+s+$ $k\tau\epsilon 0,n.$

The proof of Lemma 2.2 is referred to [1].

3. PROOF OF THEOREM 1.1 AND COROLLARY 1.2

To show $\hat{\lambda}_{c,n,k}(s)$ converges completely to λ_c, it suffices to check, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\left|\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)\right| > \varepsilon\right) < \infty. \tag{3.1}$$

The probability appearing in (3.1) can be written as

$$P\left(\left|\hat{\lambda}_{c,n,K}(s) - E\hat{\lambda}_{c,n,K}(s) + E\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)\right| > \varepsilon\right). \tag{3.2}$$

Bythe triangle inequality, the probability in (3.2) does not exceed

$$P\left(\left|\hat{\lambda}_{c,n,K}(s) - E\hat{\lambda}_{c,n,K}(s)\right| > \varepsilon - \left|E\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)\right|\right). \tag{3.3}$$

By Lemma 2.1, for sufficiently large *n*, we have that$\left|E\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)\right| \le \frac{\varepsilon}{2}$. By this argument and Chebyshev inequality, the probability in (3.3) is equal to

$$P\left(\left|\hat{\lambda}_{c,n,K}(s) - E\hat{\lambda}_{c,n,K}(s)\right| > \frac{\varepsilon}{2}\right) \le \frac{4Var\left(\hat{\lambda}_{c,n,K}(s)\right)}{\varepsilon^2}.$$

Hence, the l.h.s. of (3.1) does not exceed

$$\sum_{n=1}^{\infty} \frac{4Var\left(\hat{\lambda}_{c,n,K}(s)\right)}{\varepsilon^2}. \tag{3.4}$$

Therefore, to proof (3.1), it suffices to show that (3.4) is a convergent series.The subsequents analysis are by using Lemma 2.2.

First we consider the case $0 < b < 1$. By (2.7), the series in (3.4) can be written as

$$\sum_{n=1}^{\infty} \frac{4}{\varepsilon^2}\left\{\left[\frac{\tau\lambda_c(s)}{n^{b+1}h_n(1-b)}\int_{-1}^{1} K^2(x)dx\right] + o\left(\frac{1}{n^{b+1}h_n}\right)\right\} \tag{3.5}$$

as $n \to \infty$. Since $h_n = n^{-\alpha}$, the quantity in (3.5) can be written as

$$\sum_{n=1}^{\infty} \frac{4}{\varepsilon^2}\left\{\left[\frac{\tau\lambda_c(s)}{n^{b+1-\alpha}(1-b)}\int_{-1}^{1} K^2(x)dx\right] + o\left(\frac{1}{n^{b+1-\alpha}}\right)\right\}$$

as $n \to \infty$.

Since $0 < \alpha < b < 1$, which is equivalent to $b + 1 - \alpha > 1$, then we see that the series in (3.5) is convergent. This completes the proof of (3.4) for $0 < b < 1$.

Next we consider the case$b = 1$. By (2.8), the quantity in (3.4) can be written as

$$\sum_{n=1}^{\infty} \frac{4}{\varepsilon^2}\left\{\left(\frac{\tau\lambda_c(s)ln(n)}{n^2h_n}\int_{-1}^{1} K^2(x)dx\right) + o\left(\frac{ln(n)}{n^2h_n}\right)\right\} \tag{3.6}$$

as $n \to \infty$. Since $h_n = n^{-\alpha}$, then the quantity in (3.6) can be written as

$$\sum_{n=1}^{\infty} \frac{4}{\varepsilon^2}\left\{\left(\frac{\tau\lambda_c(s)ln(n)}{n^{2-\alpha}}\int_{-1}^{1} K^2(x)dx\right) + o\left(\frac{ln(n)}{n^{2-\alpha}}\right)\right\} \tag{3.7}$$

as $n \to \infty$. Since $0 < \alpha < 1$, we have $2 - \alpha > 1$. Hence the series in (3.7) is convergent. This completes the proof of (3.4) for $b = 1$.

Finally we consider the case$b > 1$. By (2.9), the quantity in (3.4) can be written as

$$\sum_{n=1}^{\infty} \frac{4}{\varepsilon^2}\left\{\frac{\tau^{2-b}\lambda_c(s)\zeta(b)}{n^2h_n}\int_{-1}^{1} K^2(x)dx + o\left(\frac{1}{n^2h_n}\right)\right\} \tag{3.8}$$

as $n \to \infty$. Since $h_n = n^{-\alpha}$, then the quantity in (3.8) can be written as

$$\sum_{n=1}^{\infty} \frac{4}{\varepsilon^2}\left\{\frac{\tau^{2-b}\lambda_c(s)\zeta(b)}{n^{2-\alpha}}\int_{-1}^{1} K^2(x)dx + o\left(\frac{1}{n^{2-\alpha}}\right)\right\} \tag{3.9}$$

as $n \to \infty$. Since $0 < \alpha < 1$, we have $2 - \alpha > 1$. Hence the series in (3.9) is convergent. Therefore, we have proved (3.4) for the case $b > 1$. This completes the proof of Theorem 1.1.

To show that $\hat{\lambda}_{c,n,K}(s)$ is a strong consistent estimator of $\lambda_c(s)$, it suffices to show that [11], for any $\varepsilon > 0$,

$$P\left(\lim_{n\to\infty}\left|\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)\right| \ge \varepsilon\right) = 0. \tag{3.10}$$

By Theorem 1.1, we have (3.1). By (3.1) and the Borel-Cantelli Lemma, we have that the events $\left\{\left|\hat{\lambda}_{c,n,K}(s) - \lambda_c(s)\right| > \varepsilon\right\}$ only occur at a finite many times, which implies (3.10). This completes the proof of Corollary 1.2.

4. CONCLUSION

In this paper, we have proved strong consistency properties for the estimator which proposed in [1]. The proofs were presented in Theorem 1.1 and corollary 1.2.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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