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Stress Fields and Concentrations around Circular Discontinuity: A Methodological Review with Principles of Elasticity

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Authors' contributions

This work was carried out in collaboration between both authors. Author SB performed the computational work and wrote the first draft of the manuscript. The guidance was provided by Author VGU. Both authors read and approved the final manuscript.

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ABSTRACT

A comprehensive review on the methodology to obtain two dimensional stress field around a discontinuity in the form of a circular hole in the plate subjected to various types of, uniform, axisymmetric and non-axisymmetric monotonic loads at infinity *viz*. uni-axial tensile, equal bi-axial (tensile-tensile and tensile-compressive) and pure shear is presented with the help of the basic principles of elasticity. The material of the plate is considered to be homogenous, isotropic and linear elastic. Effect of the difference in the type of far field load over the nature and the magnitude of stress fields is examined. Fundamental bi-harmonic equation involving Airy's stress function is used. The stress function, determined by assuming it in the form of trigonometric series and by employing suitable mathematical substitutions, is made to satisfy the bi-harmonic equation.

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Constants of the stress function are found from the boundary conditions. Stress concentrations at the surface of the hole and at locations away from the hole are obtained for all the investigated load cases. Criteria of failure and fatigue life estimations of the plate made of linear elastic or elastic plastic material are touched upon. Stress solutions in cases of bi-axial loads of unidentical magnitudes are also presented.

Keywords: Airy's stress function; bi-harmonic equation; circular hole; discontinuity; stress concentration.

NOTATIONS

a	*Radius of circular hole*
b	*Far field location*
c	*Fatigue ductility exponent*
d	*Fracture strength exponent*
A-D	*Constants*
E	*Modulus of elasticity*
E_S	*Secant modulus*
HCF	*High cycle fatigue*
I,J,K,L	*Points of interest at surface of hole*
*I^,J^,K^,L^*	*Points of interest away from hole*
LCF	*Low cycle fatigue*
n	*Ratio of transverse and longitudinal far field stress*
N	*Fatigue life*
$2N_f$	*Reversals to failure*
q	*Notch sensitivity index*
r	*Radial coordinate from hole centre*
R	*Stress ratio*
S	*Uniform monotonic stress at infinity*
sc	*Elastic stress concentration factor*
sc_f	*Fatigue stress concentration factor*
sc_p	*Plastic stress concentration factor*
S-N	*Cyclic stress amplitude vs cycles at R =-1*
YS	*Yield strength*
UTS	*Ultimate tensile strength*
ν	*Poisson's ratio*
ϕ	*Airy's stress function*
θ	*Angular coordinate w.r.t. hole axis*
σ_x	*Normal stress (in x direction)*
σ_y	*Normal stress (in y direction)*
σ_a	*Cyclic stress amplitude* $\left(\frac{\sigma_{max}-\sigma_{min}}{2}\right)$
σ_f'	*Fracture strength*
σ_m	*Non-zero mean stress*
σ_N	*Fatigue strength at R = -1/Design cyclic stress amplitude*
τ_{xy}	*Shear stress in x-y coordinate system*
$\varepsilon_x, \varepsilon_y$	*Normal strains in x-y coordinate system*
ε	*Total strain under monotonic load*
ε_a	*Cyclic strain amplitude* $\left(\frac{\sigma_a}{E}\right)$
$\Delta\varepsilon$	*Range of cyclic strain* ($2\varepsilon_a$)
ε_f	*Strain at fracture under monotonic load*
ε_f'	*Fatigue ductility coefficient*
γ_{xy}	*Shear strain in x-y coordinate system*
σ_r	*Radial stress (in r direction)*
σ_θ	*Tangential stress (in θ direction)*
$\tau_{r\theta}$	*Shear stress in r-θ coordinate system*

1. INTRODUCTION

An opening or a hole in the structure leading to the formation of a discontinuity due to sudden change in the geometry is well known. Discontinuity in any structure under the action of load gives rise to localised effect in the form of stress concentration [1]. Stress concentration acts as stress raiser that increases the magnitude of stress at the surface of the discontinuity and in its proximity. These highly stressed locations, where stress exceeds applied or far field stress manifold, act as favourable sites for initiation of damage that finally leads to the failure of the structure. Knowhow about the effect of discontinuity therefore assumes high practical significance in order to ensure safe operation of a discontinuous structure when pressed into service. Geometrical imperfections like sharp edges and corners in a machine part also act as discontinuities. Special attention is therefore given at the design stage of the component itself to avoid discontinuities by recommending polished surfaces and replacement of edges by fillets. Likewise, components with fabrication defects are rejected for further use. In case discontinuities like holes are unavoidable in the structure, then the permissible load over the structure is selected such that the magnitude of maximum stress parameter at critical location is less than the allowable material property. Consequently, it can be inferred that allowable load over the structure with discontinuity is less than that over the one without discontinuity or in other words the discontinuity restricts the safe load that can be applied over the structure. Since the phenomenon of stress concentration is impossible to be captured by conventional strength of material solutions, elasticity based approach is often required to accurately predict the magnitude of stress fields and concentrations around the discontinuities. However, this method also has drawbacks. Although its procedure is generalized and is same for all the problems, an appropriate stress function that satisfies the bi-harmonic equation and also at the same time fulfils the boundary conditions of a particular problem is difficult to find or guess. Besides, stress functions considerably vary from problem to problem and critically depend on the type of far field load and the body constraints. Moreover, with the advent of finite element technique that is also based on elasticity formulations but is much faster, reasonably accurate and commercially more viable, research on classical elasticity procedures has to some extent lost the momentum. Yet elasticity based solutions if available are considered to be highly accurate due to their closed form nature and are preferred over finite element solutions that being numerical in nature are approximate and at best can only be near to closed form solutions. As the result, researchers still continue to show interest in the field.

Hitherto, discontinuities in structures have been extensively studied with the principles of elasticity. Initial work began with examination of discontinuities in isotropic bodies. Kirsch [2] in 1898 obtained two dimensional stress solution around a single circular hole in homogenous and isotropic plate when loaded at infinity. Dumont [3] on similar lines determined stress concentration around an open circular hole in an infinite plate subjected to bending normal to the plane of the plate. Problems dealing with plate of finite dimensions containing multiple holes of different shapes were solved with conformal mapping by Muskhelishvili [4]. Reviews on the subject were later on performed by Savin [5] and Neuber [6] using complex variable techniques. Lekhnitskii [7] attempted series approach for similar problems. Bhargava et al. [8] prescribed normal and tangential tractions and radial coupled stress on circular boundary with complex variable approach to obtain the solution. Dhawan et al. [9] used finite element method to determine stress concentration caused by central openings of simple and complex geometry in rectangular and circular plates of finite size when subjected to tensile and compressive loads. They found rectangular plate to be better than circular one from stress point of view. They also observed that the opening of the shape of isosceles triangle resulted in lower stress concentration when compared to other geometries. Folias et al. [10] employed double fourier integral transform to Navier's equation followed by contour integrations to obtain three dimensional stress field around a circular hole in the plate of arbitrary thickness.

Numerous elasticity based investigations on stress fields around single/multiple discontinuities in homogenous or orthotropic and functionally graded material systems have also been reported. Zhang et al. [11] used Schwarz's alternating procedure and Muskhelishvili's complex variable functions to develop an iterative algorithm method for calculation of stress in an elastic solid of infinite extent containing multiple elliptic holes subjected to external loading at infinity. The algorithm was based on approximation of resultant force vector on each elliptical hole boundary as series of complex variables. Chen at al. derived [12] the null-field integral equation for a medium containing circular cavities with arbitrary radii and positions under uniformly remote shear by adopting Fourier series for boundary densities. Konish and Whitney [13] presented an approximate solution in the form of a polynomial for normal stress distribution adjacent to a circular hole in an infinite orthotropic plate. Yang et al. [14] studied two-dimensional stress distribution in a functionally graded plate with a circular hole under arbitrary constant loads. Stress distribution in the plate was derived using the method of piece-wise homogeneous layers and the theory of complex variable functions. Fan and Wu [15] examined the laminate weakened by multiple equal elliptical holes in series by treating it as an anisotropic, infinite, multiply connected thin plate. They used Faber series expansion and complex potential method to obtain stress concentration under the effect of arbitrary in-plane loads at infinity. Xu et al. [16] investigated a finite composite plate with multiple elliptical holes by employing complex potential method in plane theory of elasticity for an anisotropic body, Faber series expansion, conformal mapping and least square boundary collocation techniques in computations. The effects of plate and hole sizes, layups, relative distance between holes and total number of holes and their locations on stress distribution were obtained. Mohammadi et al. [17] using Frobenius series approach obtained stress concentration factor around a circular hole subjected to uniform biaxial tension and pure shear in an infinite plate made of functionally graded material in which both Young's modulus and Poisson's ratio varied in the radial direction.

Discontinuities have been examined by experimental and numerical methods too. Toubal et al. [18] used non-contact measurement method namely electronic speckle pattern interferometer (ESPI) to examine tensile strain field in a composite plate weakened by a circular hole. Results confirmed strain concentrations near the singularity. The experimental values were in good agreement with predictions from the theoretical model previously developed by Lekhnitskii's [7]. Rowlands et al. [19] performed finite element analysis to study anisotropic states of stress, strain and fracture of glass-epoxy plate containing a circular hole under uniaxial tension. Pan et al. [20] developed a three-dimensional boundary element method for stress analysis of a composite laminate with holes. Kubair and Chandar [21] numerically investigated the effect of material property inhomogeneity over stress concentration factor due to circular hole in functionally graded panels. Multiple isoparametric finite element formulation was used to simulate the elastostatic boundary value problem.

Most of the structures in use are made of, single, homogenous and isotropic materials. Also a lot of practical engineering problems on ground are effectively handled by two dimensional approaches, although at the expense of slight accuracy, instead of resorting to complex three dimensional procedures. This becomes possible by adopting plane stress and plane strain considerations. Among various possible shapes of discontinuities that can exist in commercial structures, circular ones are quite common because of their ease of manufacture. From the point of view of micro-mechanics of failure especially in ductile materials that are used more in comparison with brittle counterparts, circular holes assume importance because voids that originate at the interfaces between foreign particles or impurities and parent grains at ultimate tensile strengths of ductile materials are circular in shape that later on grow and assume elliptical forms before coalescing with each other under the influence of strain localisation leading to the formation of a fracture inducing crack. Keeping the above in mind, the present paper systematically reviews the methodology to obtain stress field, with the help of basic principles of elasticity, around a circular hole in an isotropic, homogenous and linear elastic plate subjected to different types of, uniform, axisymmetric and non-axisymmetric monotonic loads at infinity *viz*. uni-axial tensile, equal bi-axial (tensile-tensile and tensile-compressive) and pure shear stress in two dimensional (2D) coordinate system. All stress solutions and associated formulations are derived ab initio. Fundamental bi-harmonic equation involving Airy's stress function is used in the analysis. The stress function, determined by assuming it in the form of trigonometric series and by employing suitable mathematical substitutions, is made to satisfy the bi-harmonic equation. Constants of stress function are found from the boundary conditions. Stress solution is finally obtained from stress function that is free of unknowns. Results of stress concentration at the surface of the hole, near and at the locations away from the hole are obtained for all the investigated load cases. Variation in nature and magnitude of stress field around the hole due to different types of far field loads is illustrated and discussed. Criteria of failure under monotonic load and fatigue life estimations of the plate made of linear elastic or elastic plastic material are touched upon. Stress solutions in cases of bi-axial loads of un-identical magnitudes are also presented.

2. METHODOLOGY

The bi-harmonic equation is derived by combining compatibility, stress-strain constitutive and equilibrium equations and is same in both plane stress and plane strain conditions. Refer Appendix A. The equation in 2D cartesian (x-y) coordinate system is $\left(\frac{\partial^2}{dx^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = 0$ where ϕ is the Airy's stress function. Stress field in this coordinate system is defined as normal stresses, $\sigma_x = \frac{\partial^2\phi}{\partial y^2}$, $\sigma_y = \frac{\partial^2\phi}{\partial x^2}$ and shear stress, $\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$. Refer Appendix B. The stated stress field satisfies equilibrium conditions $\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0;\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$ and therefore the stress field is valid. Refer Appendix C. Transformation of bi-harmonic equation from cartesian to polar $(r-\theta)$ coordinate system results in $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\phi = 0$. Stress field in this coordinate system is defined as radial stress, $\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$, tangential stress, $\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}$ and shear stress $\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$. Refer Appendix D. The stated stress field satisfies equilibrium conditions $\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0;\ \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{\theta r}}{r} = 0$ and therefore the stress field is also valid.

2.1 Derivation of Airy's Function

2.1.1 Axisymmetric load

Since the magnitude of stress component is independent of θ in axisymmetric load i.e. $\left(\frac{\partial}{\partial \theta} = 0\right)$, bi-harmonic equation in polar coordinate system in such a case reduces to

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\phi = 0 \quad \text{or} \quad \frac{\partial^4 \phi}{\partial r^4} + \frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial \phi}{\partial r}\right) + \frac{1}{r}\frac{\partial^3 \phi}{\partial r^3} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial r}\right) = 0 \tag{1}$$

Simplification of second and fourth terms of Eq. (1) leads to:-II.

$$\frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\frac{\partial \phi}{\partial r}\right) = \frac{\partial}{\partial r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial r}\right) = \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r^2}\frac{\partial \phi}{\partial r}\right] = \frac{1}{r}\frac{\partial^3 \phi}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r^3}\frac{\partial \phi}{\partial r} \tag{2}$$

$$\text{IV.}\ \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial r}\right) = \frac{1}{r}\left[\frac{1}{r}\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r^2}\frac{\partial \phi}{\partial r}\right] = \frac{1}{r^2}\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r^3}\frac{\partial \phi}{\partial r} \tag{3}$$

Using Eq. (2) and (3) in Eq. (1) results in

$$\frac{\partial^4 \phi}{\partial r^4} + \frac{2}{r}\frac{\partial^3 \phi}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r^3}\frac{\partial \phi}{\partial r} = 0 \tag{4}$$

On using mathematical substitutions, $r = e^t$, $dr = e^t dt$, $\log_e r = t$, and employing chain rule of differentiation, we obtain the following

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial t}\frac{\partial t}{\partial r} = \frac{1}{e^t}\frac{\partial \phi}{\partial t} \tag{5}$$

$$\frac{\partial^2 \phi}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{1}{e^t}\frac{\partial \phi}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{1}{e^t}\frac{\partial \phi}{\partial t}\right)\frac{\partial t}{\partial r} = \frac{1}{e^t}\left[\frac{1}{e^t}\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{e^t}\frac{\partial \phi}{\partial t}\right] = \frac{1}{e^{2t}}\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{e^{2t}}\frac{\partial \phi}{\partial t}$$

$$\frac{\partial^3 \phi}{\partial r^3} = \frac{\partial}{\partial r}\left[\frac{1}{e^{2t}}\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{e^{2t}}\frac{\partial \phi}{\partial t}\right] = \frac{\partial}{\partial t}\left[\frac{1}{e^{2t}}\frac{\partial^2 \phi}{\partial t^2}\right]\frac{\partial t}{\partial r} - \frac{\partial}{\partial t}\left[\frac{1}{e^{2t}}\frac{\partial \phi}{\partial t}\right]\frac{\partial t}{\partial r} \tag{6}$$

$$= \left[\frac{1}{e^{2t}}\frac{\partial^3\phi}{\partial t^3} - \frac{2}{e^{2t}}\frac{\partial^2\phi}{\partial t^2}\right]\frac{1}{e^t} - \left[\frac{1}{e^{2t}}\frac{\partial^2\phi}{\partial t^2} - \frac{2}{e^{2t}}\frac{\partial\phi}{\partial t}\right]\frac{1}{e^t} = \frac{1}{e^{3t}}\frac{\partial^3\phi}{\partial t^3} - \frac{3}{e^{3t}}\frac{\partial^2\phi}{\partial t^2} + \frac{2}{e^{3t}}\frac{\partial\phi}{\partial t} \quad (7)$$

$$\frac{\partial^4\phi}{\partial r^4} = \frac{\partial}{\partial r}\left(\frac{\partial^3\phi}{\partial r^3}\right) = \frac{\partial}{\partial r}\left[\frac{1}{e^{3t}}\frac{\partial^3\phi}{\partial t^3} - \frac{3}{e^{3t}}\frac{\partial^2\phi}{\partial t^2} + \frac{2}{e^{3t}}\frac{\partial\phi}{\partial t}\right]$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{e^{3t}}\frac{\partial^3\phi}{\partial t^3}\right]\frac{\partial t}{\partial r} - \frac{\partial}{\partial t}\left[\frac{3}{e^{3t}}\frac{\partial^2\phi}{\partial t^2}\right]\frac{\partial t}{\partial r} + \frac{\partial}{\partial t}\left[\frac{2}{e^{3t}}\frac{\partial\phi}{\partial t}\right]\frac{\partial t}{\partial r}$$

$$= \left[\frac{1}{e^{3t}}\frac{\partial^4\phi}{\partial t^4} - \frac{3}{e^{3t}}\frac{\partial^3\phi}{\partial t^3}\right]\frac{1}{e^t} - \left[\frac{3}{e^{3t}}\frac{\partial^3\phi}{\partial t^3} - \frac{9}{e^{3t}}\frac{\partial^2\phi}{\partial t^2}\right]\frac{1}{e^t} + \left[\frac{2}{e^{3t}}\frac{\partial^2\phi}{\partial t^2} - \frac{6}{e^{3t}}\frac{\partial\phi}{\partial t}\right]\frac{1}{e^t}$$

$$= \frac{1}{e^{4t}}\frac{\partial^4\phi}{\partial t^4} - \frac{6}{e^{4t}}\frac{\partial^3\phi}{\partial t^3} + \frac{11}{e^{4t}}\frac{\partial^2\phi}{\partial t^2} - \frac{6}{e^{4t}}\frac{\partial\phi}{\partial t} \quad (8)$$

Substitution of Eq. (5) to Eq. (8) along with $r = e^t$ in Eq. (4) gives

$$\frac{\partial^4\phi}{\partial t^4} - 4\frac{\partial^3\phi}{\partial t^3} + 4\frac{\partial^2\phi}{\partial t^2} = 0 \quad (9)$$

Use of operator, $D = \frac{\partial}{\partial t}$, in Eq. (9) results in $\left[D^4 - 4D^3 + 4D^2\right]\phi = 0$. Roots of complimentary function are D = 0, 0, +2, +2.

Therefore, $\phi = Ae^{0t} + Bte^{0t} + Ce^{2t} + Dte^{2t} = A + B\log_e r + Cr^2 + Dr^2\log_e r$.

Stress solution when $\frac{\partial}{\partial\theta} = 0$ is

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} = \frac{1}{r}\left[\frac{B}{r} + 2Cr + 2Dr\log_e r + Dr\right] = \frac{B}{r^2} + 2C + 2D\log_e r + D = \frac{B}{r^2} + 2C + D(1 + 2\log_e r) \quad (10)$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} = \frac{\partial}{\partial r}\left[\frac{B}{r} + 2Cr + 2Dr\log_e r + Dr\right] = -\frac{B}{r^2} + 2C + 2D\log_e r + 3D = -\frac{B}{r^2} + 2C + D(3 + 2\log_e r) \quad (11)$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 0 \quad (12)$$

2.1.2 Non-axisymmetric load

Bi-harmonic equation in non-axisymmetric load is rewritten as

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = 0 \quad (13)$$

Two types of stress functions, ϕ, are assumed in the form of trigonometric series for such loads. They are discussed one by one as follows:-

i). $\phi = f(r)\cos 2\theta$. Its use in Eq. (13) results in $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{4f}{r^2}\right)\cos 2\theta = 0$ (14)

On considering differential terms of L.H.S. of Eq. (14) one by one for simplification, we have

I. $\frac{\partial^2}{\partial r^2}\left(\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}-\frac{4f}{r^2}\right)\cos 2\theta=\left[\frac{\partial^4 f}{\partial r^4}+\frac{\partial}{\partial r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial r}\right)-\frac{\partial}{\partial r}\frac{\partial}{\partial r}\left(\frac{4f}{r^2}\right)\right]\cos 2\theta$

$$=\left\{\frac{\partial^4 f}{\partial r^4}+\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial^2 f}{\partial r^2}-\frac{1}{r^2}\frac{\partial f}{\partial r}\right]-\frac{\partial}{\partial r}\left[\frac{4}{r^2}\frac{\partial f}{\partial r}-\frac{8}{r^3}f\right]\right\}\cos 2\theta$$

$$=\left\{\frac{\partial^4 f}{\partial r^4}+\frac{1}{r}\frac{\partial^3 f}{\partial r^3}-\frac{1}{r^2}\frac{\partial^2 f}{\partial r^2}-\frac{1}{r^2}\frac{\partial^2 f}{\partial r^2}+\frac{2}{r^3}\frac{\partial f}{\partial r}-\frac{4}{r^2}\frac{\partial^2 f}{\partial r^2}+\frac{8}{r^3}\frac{\partial f}{\partial r}+\frac{8}{r^3}\frac{\partial f}{\partial r}-\frac{24f}{r^4}\right\}\cos 2\theta$$

$$=\left\{\frac{\partial^4 f}{\partial r^4}+\frac{1}{r}\frac{\partial^3 f}{\partial r^3}-\frac{6}{r^2}\frac{\partial^2 f}{\partial r^2}+\frac{18}{r^3}\frac{\partial f}{\partial r}-\frac{24f}{r^4}\right\}\cos 2\theta \tag{15}$$

II. $\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}-\frac{4f}{r^2}\right)\cos 2\theta=\left\{\frac{1}{r}\frac{\partial^3 f}{\partial r^3}+\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial f}{\partial r}\right]-\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{4f}{r^2}\right]\right\}\cos 2\theta$

$$=\left\{\frac{1}{r}\frac{\partial^3 f}{\partial r^3}+\frac{1}{r}\left[\frac{1}{r}\frac{\partial^2 f}{\partial r^2}-\frac{1}{r^2}\frac{\partial f}{\partial r}\right]-\frac{1}{r}\left[\frac{4}{r^2}\frac{\partial f}{\partial r}-\frac{8f}{r^3}\right]\right\}\cos 2\theta$$

$$=\left\{\frac{1}{r}\frac{\partial^3 f}{\partial r^3}+\frac{1}{r^2}\frac{\partial^2 f}{\partial r^2}-\frac{5}{r^3}\frac{\partial f}{\partial r}+\frac{8f}{r^4}\right\}\cos 2\theta \tag{16}$$

III. $\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}-\frac{4f}{r^2}\right)\cos 2\theta$

$$=\left\{\frac{1}{r^2}\frac{\partial^2 f}{\partial r^2}\frac{\partial^2}{\partial \theta^2}(\cos 2\theta)+\frac{1}{r^3}\frac{\partial f}{\partial r}\frac{\partial^2}{\partial \theta^2}(\cos 2\theta)-\frac{4f}{r^4}\frac{\partial^2}{\partial \theta^2}(\cos 2\theta)\right\}$$

$$=\left\{-\frac{4}{r^2}\frac{\partial^2 f}{\partial r^2}-\frac{4}{r^3}\frac{\partial f}{\partial r}+\frac{16f}{r^4}\right\}\cos 2\theta \tag{17}$$

Substitution of Eq. (15) to Eq. (17) in Eq. (14) provides

$$\left(\frac{\partial^4 f}{\partial r^4}+\frac{2}{r}\frac{\partial^3 f}{\partial r^3}-\frac{9}{r^2}\frac{\partial^2 f}{\partial r^2}+\frac{9}{r^3}\frac{\partial f}{\partial r}\right)\cos 2\theta=0 \text{ or } r^4\frac{\partial^4 f}{\partial r^4}+2r^3\frac{\partial^3 f}{\partial r^3}-9r^2\frac{\partial^2 f}{\partial r^2}+9r\frac{\partial f}{\partial r}=0 \tag{18}$$

On assuming, $f=r^m$, we obtain

$$\frac{\partial f}{\partial r}=mr^{m-1},\frac{\partial^2 f}{\partial r^2}=m(m-1)r^{m-2},\frac{\partial^3 f}{\partial r^3}=m(m-1)(m-2)r^{m-3},$$
$$\frac{\partial^4 f}{\partial r^4}=m(m-1)(m-2)(m-3)r^{m-4}$$

Substitution of above in Eq. (18) gives

$$[m(m-1)(m-2)(m-3)+2m(m-1)(m-2)-9m(m-1)+9m]r^m=0 \text{ or } m^4-4m^3-4m^2+16m=0 \tag{19}$$

Roots of Eq. (19) are m = 2, 4, -2 and 0

Therefore, $f = Ar^2 + Br^4 + Cr^{-2} + Dr^0 = Ar^2 + Br^4 + \frac{C}{r^2} + D$ and $\phi = \left(Ar^2 + Br^4 + \frac{C}{r^2} + D\right)\cos 2\theta$

Stress solution is

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = \frac{1}{r}\left(2Ar + 4Br^3 - \frac{2C}{r^3}\right)\cos 2\theta - \frac{4}{r^2}\left(Ar^2 + Br^4 + \frac{C}{r^2} + D\right)\cos 2\theta = \left(-2A - \frac{6C}{r^4} - \frac{4D}{r^2}\right)\cos 2\theta \quad (20)$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial \phi}{\partial r}\right) = \frac{\partial}{\partial r}\left(2Ar + 4Br^3 - \frac{2C}{r^3}\right)\cos 2\theta = \left(2A + 12Br^2 + \frac{6C}{r^4}\right)\cos 2\theta \quad (21)$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = -\frac{\partial}{\partial r}\left[-2\left(Ar + Br^3 + \frac{C}{r^3} + \frac{D}{r}\right)\sin 2\theta\right] = \left(2A + 6Br^2 - \frac{6C}{r^4} - \frac{2D}{r^2}\right)\sin 2\theta \quad (22)$$

ii). $\phi = f(r)\sin 2\theta$. Its use in Eq. (13) results in $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{4f}{r^2}\right)\sin 2\theta = 0$ (23)

On considering differential terms one by one and repeating the procedure stated in Section b.1., we have

$$\text{I. } \frac{\partial^2}{\partial r^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{4f}{r^2}\right)\sin 2\theta = \left\{\frac{\partial^4 f}{\partial r^4} + \frac{1}{r}\frac{\partial^3 f}{\partial r^3} - \frac{6}{r^2}\frac{\partial^2 f}{\partial r^2} + \frac{18}{r^3}\frac{\partial f}{\partial r} - \frac{24f}{r^4}\right\}\sin 2\theta \quad (24)$$

$$\text{II. } \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{4f}{r^2}\right)\sin 2\theta = \left\{\frac{1}{r}\frac{\partial^3 f}{\partial r^3} + \frac{1}{r^2}\frac{\partial^2 f}{\partial r^2} - \frac{5}{r^3}\frac{\partial f}{\partial r} + \frac{8f}{r^4}\right\}\sin 2\theta \quad (25)$$

$$\text{III. } \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{4f}{r^2}\right)\sin 2\theta = \left\{-\frac{4}{r^2}\frac{\partial^2 f}{\partial r^2} - \frac{4}{r^3}\frac{\partial f}{\partial r} + \frac{16f}{r^4}\right\}\sin 2\theta \quad (26)$$

Substitution of Eq. (24) to Eq. (26) in Eq. (23) gives $\left(\frac{\partial^4 f}{\partial r^4} + \frac{2}{r}\frac{\partial^3 f}{\partial r^3} - \frac{9}{r^2}\frac{\partial^2 f}{\partial r^2} + \frac{9}{r^3}\frac{\partial f}{\partial r}\right)\sin 2\theta = 0$ or

$$r^4\frac{\partial^4 f}{\partial r^4} + 2r^3\frac{\partial^3 f}{\partial r^3} - 9r^2\frac{\partial^2 f}{\partial r^2} + 9r\frac{\partial f}{\partial r} = 0 \quad (27)$$

Eq. (27) is same as Eq. (18). Therefore, $f = Ar^2 + Br^4 + Cr^{-2} + Dr^0 = Ar^2 + Br^4 + \frac{C}{r^2} + D$ and $\phi = \left(Ar^2 + Br^4 + \frac{C}{r^2} + D\right)\sin 2\theta$. However stress solution differs and is as follows:-

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = \frac{1}{r}\left(2Ar + 4Br^3 - \frac{2C}{r^3}\right)\sin 2\theta - \frac{4}{r^2}\left(Ar^2 + Br^4 + \frac{C}{r^2} + D\right)\sin 2\theta = \left(-2A - \frac{6C}{r^4} - \frac{4D}{r^2}\right)\sin 2\theta \quad (28)$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = \frac{\partial}{\partial r}\left(2Ar + 4Br^3 - \frac{2C}{r^3}\right)\sin 2\theta = \left(2A + 12Br^2 + \frac{6C}{r^4}\right)\sin 2\theta \quad (29)$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = -\frac{\partial}{\partial r}\left[2\left(Ar + Br^3 + \frac{C}{r^3} + \frac{D}{r}\right)\cos 2\theta\right] = \left(-2A - 6Br^2 + \frac{6C}{r^4} + \frac{2D}{r^2}\right)\cos 2\theta \qquad (30)$$

3. LOAD CASES

A plate of large planer dimensions containing a circular hole of radius, *a*, is considered under the following types of, uniform, far field monotonic stress, *S*, (at infinity) to obtain 2D stress state at any point of co-ordinates r and θ in the domain. The solutions are valid from the surface of the hole where $\frac{r}{a} = 1$ to far field locations where $\frac{r}{a} = \infty$ or $\frac{a}{r} = 0$.

3.1 Uni-axial Tensile

Refer Fig. 1a). Tensile stress at infinity (say at *r* = *b* where *b* >> *a*) is $\sigma_x = S, \sigma_y = 0, \tau_{xy} = 0$ in cartesian coordinate system. On using transformation equations for obtaining far field stress field in polar coordinate system with known stress state in cartesian coordinate system, we have

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\sin 2\theta = S\cos^2\theta = \frac{S}{2} + \frac{S}{2}\cos 2\theta$$
$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - \tau_{xy}\sin 2\theta = S\sin^2\theta = \frac{S}{2} - \frac{S}{2}\cos 2\theta$$
$$\tau_{r\theta} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}\cos 2\theta = -\frac{S}{2}\sin 2\theta \qquad (31)$$

Far field stress can be split into axisymmetric and non-axisymmetric components as under that act as boundary conditions:-

a) $\sigma_r = \frac{S}{2}; \sigma_\theta = \frac{S}{2}; \tau_{r\theta} = 0$ (Axisymmetric)

b) $\sigma_r = \frac{S}{2}\cos 2\theta; \sigma_\theta = -\frac{S}{2}\cos 2\theta; \tau_{r\theta} = -\frac{S}{2}\sin 2\theta$ (Non-axisymmetric)

3.1.1 Solution for axisymmetric load at a)

Stress solution for such load is given by Eq. (10) to Eq. (12). At the centre of the hole, *r = 0*, $\log_e r = \infty$ that makes the solution infinite which is not true. Hence constant *D* has to be equal to 0. Stress solution then assumes the following form, $\sigma_r = \frac{B}{r^2} + 2C$; $\sigma_\theta = -\frac{B}{r^2} + 2C$; $\tau_{r\theta} = 0$. The solution involves two unknown constants, *B* and *C*. Therefore two boundary conditions are required. Using previously stated boundary conditions i) $\sigma_r = \frac{S}{2}$ at *r* = *b* and ii) $\sigma_\theta = \frac{S}{2}$ at *r* = *b* results in $\frac{S}{2} = \frac{B}{b^2} + 2C$ and $\frac{S}{2} = -\frac{B}{b^2} + 2C$ that are contradictory to each other. Hence new boundary conditions i) $\sigma_r = \frac{S}{2}$ at *r* = *b* ii) $\sigma_r = 0$ at *r* = *a* due to free surface of the hole are tried. The condition ii) also takes into account the presence of the hole. They result in $\frac{S}{2} = \frac{B}{b^2} + 2C$; $0 = \frac{B}{a^2} + 2C$, solution of which gives

$$B = \frac{a^2 b^2 S}{2(a^2 - b^2)} = \frac{a^2 S}{2\left(\frac{a^2}{b^2} - 1\right)} = -\frac{a^2 S}{2} \text{ since } b >> a$$

or $\frac{a}{b} \approx 0$; $C = -\frac{B}{2a^2} = \frac{S}{4}$

Stress solution without constants is

$$\sigma_r = -\frac{a^2 S}{2r^2} + \frac{S}{2}; \sigma_\theta = \frac{a^2 S}{2r^2} + \frac{S}{2}; \tau_{r\theta} = 0 \qquad (32)$$

The condition, $\sigma_\theta = \frac{S}{2}$ at *r* = *b*, is satisfied in the solution. Hence the solution is valid.

3.1.2 Solution for non-axisymmetric load at b)

Stress function, $\phi = f(r)\cos 2\theta$, is tried for which stress solution is given by Eq. (20) to Eq. (22). The solution involves four constants *A-D*. Use of the following four boundary conditions

i) $\sigma_r = \frac{S}{2}\cos 2\theta$ at $r = b$ ii) $\sigma_\theta = -\frac{S}{2}\cos 2\theta$ at $r = b$ iii) $\tau_{r\theta} = -\frac{S}{2}\sin 2\theta$ at $r = b$ iv) $\sigma_r = 0$ at $r = a$ results in

$$\left(-2A-\frac{6C}{b^4}-\frac{4D}{b^2}\right)=\frac{S}{2};\left(2A+12Bb^2+\frac{6C}{b^4}\right)=-\frac{S}{2};\ \left(2A+6Bb^2-\frac{6C}{b^4}-\frac{2D}{b^2}\right)=-\frac{S}{2}\ \ \left(-2A-\frac{6C}{a^4}-\frac{4D}{a^2}\right)=0 \quad (33)$$

Since variable θ does not exist in Eq. (33), selection of stress function, $\phi = f(r)\cos 2\theta$, is justified. For $\frac{a}{b} \approx 0$, the solution provides constants as $A = -\frac{S}{4}, B = 0, C = -\frac{a^4 S}{4}, D = \frac{a^2 S}{2}$. Stress solution without constants is

$$\sigma_r = \left(\frac{S}{2}+\frac{6a^4S}{4r^4}-\frac{2a^2S}{r^2}\right)\cos 2\theta\ ;\ \sigma_\theta = -\left(\frac{S}{2}+\frac{6a^4S}{4r^4}\right)\cos 2\theta\ \ \tau_{r\theta} = \left(-\frac{S}{2}+\frac{6a^4S}{4r^4}-\frac{a^2S}{r^2}\right)\sin 2\theta \quad (34)$$

Final stress solution is obtained by superimposing individual solutions of case a) and case b) at Eq. (32) and Eq. (34) respectively and is as follows:-

$$\sigma_r = \left(-\frac{a^2S}{2r^2}+\frac{S}{2}\right)+\left(\frac{S}{2}+\frac{6a^4S}{4r^4}-\frac{2a^2S}{r^2}\right)\cos 2\theta\ ;\ \sigma_\theta = \left(\frac{a^2S}{2r^2}+\frac{S}{2}\right)-\left(\frac{S}{2}+\frac{6a^4S}{4r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = \left(-\frac{S}{2}+\frac{6a^4S}{4r^4}-\frac{a^2S}{r^2}\right)\sin 2\theta \quad (35)$$

At $\frac{r}{a}=1, \sigma_r = 0$, $\sigma_\theta = S - 2S\cos 2\theta$, $\tau_{r\theta} = 0$

At $\frac{r}{a}=\infty$, $\sigma_r = \frac{S}{2}+\frac{S}{2}\cos 2\theta$, $\sigma_\theta = \frac{S}{2}-\frac{S}{2}\cos 2\theta$, $\tau_{r\theta} = -\frac{S}{2}\sin 2\theta$

3.2 Bi-axial (Tensile-tensile)

Refer Fig. 1b). Equal and bi-axial tensile stress at infinity (r = b) is $\sigma_x = S, \sigma_y = S, \tau_{xy} = 0$. On using transformation equations, we have

$$\begin{aligned}\sigma_r &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\sin 2\theta = S\cos^2\theta + S\sin^2\theta = S\\ \sigma_\theta &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - \tau_{xy}\sin 2\theta = S\sin^2\theta + S\cos^2\theta = S\\ \tau_{r\theta} &= (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}\cos 2\theta = 0\end{aligned} \quad (36)$$

This is an axisymmetric case for which the stress solution is given by Eq. (10) to Eq. (12). On adopting D = 0 as in Section 3.1 a), we have $\sigma_r = \frac{B}{r^2}+2C$; $\sigma_\theta = -\frac{B}{r^2}+2C$; $\tau_{r\theta} = 0$. Using the boundary conditions i) $\sigma_r = S$ at r = b and ii) $\sigma_r = 0$ at r = a results in

$$S = \frac{B}{b^2}+2C\ ;\ 0 = \frac{B}{a^2}+2C \quad (37)$$

Solution of Equations at (37) provides constants as

$B = \frac{a^2b^2S}{(a^2-b^2)} = -a^2S$ for $b \gg a$; $C = -\frac{B}{2a^2} = \frac{S}{2}$

Stress solution without constants is

$$\sigma_r = -\frac{a^2S}{r^2}+S\ ;\sigma_\theta = \frac{a^2S}{r^2}+S\ ;\tau_{r\theta} = 0 \quad (38)$$

The condition, $\sigma_\theta = S$ at r = b, is satisfied in the solution.

At $\frac{r}{a}=1$, $\sigma_r = 0$, $\sigma_\theta = 2S$, $\tau_{r\theta} = 0$.

At $\frac{r}{a}=\infty$, $\sigma_r = S$, $\sigma_\theta = S$, $\tau_{r\theta} = 0$.

3.3 Bi-axial (Tensile-compressive)

Refer Fig. 1c). Equal and bi-axial stress state with opposite sense at infinity (r = b) is $\sigma_x = S, \sigma_y = -S, \tau_{xy} = 0$ .On using transformation equations, we have

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\sin 2\theta = S\cos^2\theta - S\sin^2\theta = S\cos 2\theta$$
$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - \tau_{xy}\sin 2\theta = S\sin^2\theta - S\cos^2\theta = -S\cos 2\theta \quad (39)$$
$$\tau_{r\theta} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}\cos 2\theta = (-S - S)\sin\theta\cos\theta = -S\sin 2\theta$$

This is a non-axisymmetric case for which stress function is $\phi = f(r)\cos 2\theta$ is tried. Use of the following boundary conditions

i) $\sigma_r = S\cos 2\theta$ at $r = b$ ii) $\sigma_\theta = -S\cos 2\theta$ at $r = b$ iii) $\tau_{r\theta} = -S\sin 2\theta$ at $r = b$ iv) $\sigma_r = 0$ at $r = a$

in the stress solution given by Eq. (20) to Eq. (22) results in following equations which are free of variable, θ.

$$\left(-2A - \frac{6C}{b^4} - \frac{4D}{b^2}\right) = S;\ \left(2A + 12Bb^2 + \frac{6C}{b^4}\right) = -S;\ \left(2A + 6Bb^2 - \frac{6C}{b^4} - \frac{2D}{b^2}\right) = -S\ \left(-2A - \frac{6C}{a^4} - \frac{4D}{a^2}\right) = 0 \quad (40)$$

Solution of Equations at (40) for $\frac{a}{b} \approx 0$ results in constants as $A = -\frac{S}{2}, B = 0, C = -\frac{a^4 S}{2}, D = a^2 S$

Stress solution without constants is

$$\sigma_r = \left(S + \frac{6a^4 S}{2r^4} - \frac{4a^2 S}{r^2}\right)\cos 2\theta\ ;\sigma_\theta = -\left(S + \frac{6a^4 S}{2r^4}\right)\cos 2\theta\ \ \tau_{r\theta} = \left(-S + \frac{6a^4 S}{2r^4} - \frac{2a^2 S}{r^2}\right)\sin 2\theta \quad (41)$$

At $\frac{r}{a} = 1$, $\sigma_r = 0$, $\sigma_\theta = -4S\cos 2\theta$, $\tau_{r\theta} = 0$

At $\frac{r}{a} = \infty$, $\sigma_r = S\cos 2\theta$, $\sigma_\theta = -S\cos 2\theta$, $\tau_{r\theta} = -S\sin 2\theta$

3.4 Shear

Refer Fig. 1d). Shear stress at infinity (*r* = *b*) is $\sigma_x = 0, \sigma_y = 0, \tau_{xy} = S$. On using transformation equations we have

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\sin 2\theta = S\sin 2\theta$$
$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - \tau_{xy}\sin 2\theta = -S\sin 2\theta \quad (42)$$
$$\tau_{r\theta} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}\cos 2\theta = S\cos 2\theta$$

Stress function, $\phi = f(r)\cos 2\theta$, is again tried the stress solution for which is given by Eq. (20) to Eq. (22). Use of the following boundary conditions

i) $\sigma_r = S\sin 2\theta$ at $r = b$ ii) $\sigma_\theta = -S\sin 2\theta$ at $r = b$ iii) $\tau_{r\theta} = S\cos 2\theta$ at $r = b$ iv) $\sigma_r = 0$ at $r = a$ results in

$$\left(-2A - \frac{6C}{b^4} - \frac{4D}{b^2}\right) = S\tan 2\theta;\left(2A + 12Bb^2 + \frac{6C}{b^4}\right) = -S\tan 2\theta\ \left(2A + 6Bb^2 - \frac{6C}{b^4} - \frac{2D}{b^2}\right) = S\cot 2\theta;$$
$$\left(-2A - \frac{6C}{a^4} - \frac{4D}{a^2}\right) = 0 \quad (43)$$

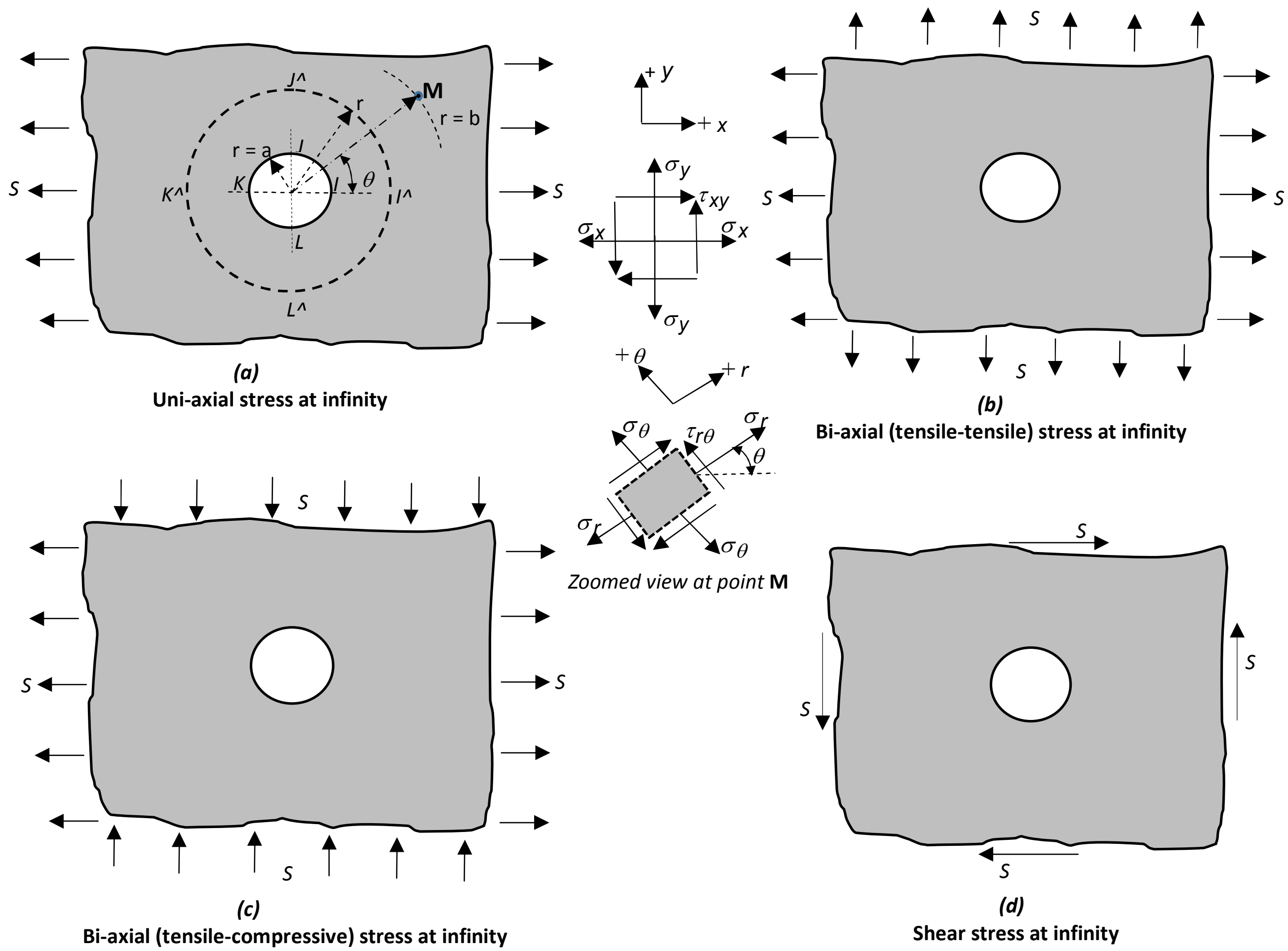


Fig. 1. Types of investigated loads

Since Eq. (43) involve variable θ, constants *A-D* that are independent of θ cannot be obtained. Hence stress function, $\phi = f(r)\cos 2\theta$, is invalid in this case. Therefore stress function, $\phi = f(r)\sin 2\theta$, is chosen for which stress solution is given by Eq. (28) to Eq. (30). Using stated boundary conditions results in the following equations that are independent of θ

$$\left(-2A - \frac{6C}{b^4} - \frac{4D}{b^2}\right) = S \,;\, \left(2A + 12Bb^2 + \frac{6C}{b^4}\right) = -S \,;\, \left(-2A - 6Bb^2 + \frac{6C}{b^4} + \frac{2D}{b^2}\right) = S \;\; \left(-2A - \frac{6C}{a^4} - \frac{4D}{a^2}\right) = 0 \qquad (44)$$

Eq. (44) are same as Eq. (40). The constants already obtained are $A = -\frac{S}{2}, B = 0, C = -\frac{a^4 S}{2}, D = a^2 S$

Stress solution without constants is

$$\sigma_r = \left(S + \frac{6a^4 S}{2r^4} - \frac{4a^2 S}{r^2}\right)\sin 2\theta \,;\, \sigma_\theta = -\left(S + \frac{6a^4 S}{2r^4}\right)\sin 2\theta \,;\, \tau_{r\theta} = \left(S - \frac{6a^4 S}{2r^4} + \frac{2a^2 S}{r^2}\right)\cos 2\theta \qquad (45)$$

At $\frac{r}{a} = 1,\ \sigma_r = 0,\ \sigma_\theta = -4S\sin 2\theta,\ \tau_{r\theta} = 0$

At $\frac{r}{a} = \infty,\ \sigma_r = S\sin 2\theta\ \ \sigma_\theta = -S\sin 2\theta, \tau_{r\theta} = S\cos 2\theta$

4. RESULTS AND DISCUSSION

Elastic stress concentrations denoted by $sc(\sigma_r), sc(\sigma_\theta), sc(\tau_{r\theta})$ are defined as $\frac{\sigma_r}{S}, \frac{\sigma_\theta}{S}$ and $\frac{\tau_{r\theta}}{S}$ in case of radial, tangential and shear stress respectively. Positive values indicate tensile while negative values denote compressive stress. Key points of interest are I, J, K and L at the surface of the hole and I^, J^, K^ and L^ at other radii. $\sigma_r = \sigma_x, \sigma_\theta = \sigma_y, \tau_{r\theta} = \tau_{xy}$ at points I, I^ $(\theta = 0\,\text{deg.})$ and K, K^ $(\theta = 180\,\text{deg.})$ whereas $\sigma_r = \sigma_y, \sigma_\theta = \sigma_x, \tau_{r\theta} = \tau_{yx}$ at points J,J^ $(\theta = 90\,\text{deg.})$ and L,L^ $(\theta = 270\,\text{deg.})$. Magnitude of stress concentration is obtained at *r = a, r = 1.5a, r = 2a, r = 3a, r = 4a, r = 5a* and *r = 6a* in each load case with the help of presented stress solutions in Section 3. *r = a* represents the locations at the surface of the hole, *r =1.5a* adjacent to the hole and *r = 6a* away from the hole.

Refer Fig. 2a) for results of Case 3.1. Tangential stress, σ_θ, fluctuates between negative and positive values at all radii. At the surface of the hole, it is compressive at I and K with its value equal to the applied stress while it is tensile at J and L with its value three times the applied stress. σ_θ drops at positions away from the surface of the hole with compressive state changing to tensile state, its magnitude finally approaching zero at I and K and applied stress at J and L. Radial stresses, σ_r, is zero at all the angles on the surface of the hole and is tensile at all other radii. Adjacent to the hole, σ_r is lower at I^ and K^ and higher at J^ and L^. Away from the hole, the trend reverses with the value higher at I^ and K^ and lesser at J^ and L^. σ_r finally approaches applied stress at I^ and K^ and zero at J^ and L^. Shear stress, $\tau_{r\theta}$, is also zero at all angles on the surface of the hole, fluctuates between negative and positive values at other radii with the values zero at I^, J^, K^ and L^ at each radius. It finally approaches 0.5 times the value of applied stress away from the hole, compressive at θ of 45 deg. and 225 deg. and tensile at θ of 135 deg. and 315 deg.

Refer Fig. 2b) for results of Case 3.2. Being an axisymmetric case, σ_θ and σ_r are tensile and equal at all the angles on a particular radius while $\tau_{r\theta}$ is zero throughout the domain. At the surface of the hole, σ_θ is twice the applied stress while σ_r is zero. σ_θ drops while σ_r increases at locations away from the hole with both finally approaching the value of applied stress.

Refer Fig. 2c) for results of Case 3.3. Tangential stress, σ_θ, fluctuates between negative and positive values at all radii. At the surface of the hole, it is compressive at I and K with its value four times the applied stress while it is tensile at J and L with its value again four times the applied stress. Magnitude of σ_θ drops at locations away from the surface of the hole and finally approaches the value of applied stress, compressive at I^ and K^ and tensile at J^ and L^. Radial stresses, σ_r , is zero at all the angles on the surface of the hole and fluctuates between positive and negative values at other radii. Adjacent to the hole, σ_r is compressive at I^ and K^ and tensile at J^ and L^. Away from the hole, the trend reverses, σ_r being tensile at I^ and K^ and compressive at J^ and L^ with the magnitudes finally approaching the value of applied stress. Shear stress, $\tau_{r\theta}$, is zero at all the angles on the surface of the hole and fluctuates between negative and positive values at all other radii with the values at I^, J^, K^ and L^ equal to zero at each radii. Far from the hole, $\tau_{r\theta}$ approaches the value of applied stress, compressive at θ of 45 deg. and 225 deg. and tensile at θ of 135 deg. and 315 deg.

Refer Fig. 2d) for results of Case 3.4. Tangential stress, σ_θ, fluctuates between negative and positive values at all radii with its value equal to zero at I,I^ J,J^, K,K^ and L,L^. Its magnitude is maximum at the surface of hole, four times the applied stress, compressive at θ of 45 deg. and 225 deg. and tensile at θ of 135 deg. and 315 deg. It approaches applied stress, away from the hole, in similar form at stated angles. Radial stress, σ_r , is zero at all the angles on the surface of the hole, fluctuates between positive and negative values at other radii with values at I^, J^, K^ and L^ equal to zero at each radius. Its trend away from the hole reverses to that at adjacent to the hole with the value approaching applied stress, tensile at θ of 45 deg. and 225 deg. and compressive at θ of 135 deg. and 315 deg. Shear stress, $\tau_{r\theta}$, also is zero at all the angles on the surface of the hole and fluctuates between negative and positive values at all radii. The value finally approaches applied stress at locations away from the hole, tensile at I^ and K^ and compressive at J^ and L^.

4.1 Failure Criteria and Fatigue Life Estimations

Maximum stress that is tangential in nature develops at the surface of the hole in all the discussed cases. The critical points with tensile stress are J and L in Case 3.1, all the points including I, J, K and L in Case 3.2, points J and L in Case 3.3 and points at angles of 135 deg. and 315 deg. in Case 3.4. Since the shear stress is zero at critical points in all the cases, tangential stress acts as the principal stress. Other two principal stresses, one of them being the radial stress, are equal to zero. Conditions for safe design at stated critical points under monotonic loads along with estimations of fatigue life of plate in HCF and LCF regimes are summarised as follows:-

4.1.1 Monotonic load

4.1.1.1 Brittle material (linear elastic)

Using Rankine's theory i.e. maximum principal stress should be less than the ultimate tensile strength of the material.

Case 3.1: $3S < UTS$, Case 3.2: $2S < UTS$, Case 3.3: $4S < UTS$, Case 3.4: $4S < UTS$

4.1.1.2 Ductile material (elastic-plastic)

Two cases exist:-

a) When stress field at the critical points is in elastic state i.e. $(sc \times S) < YS$.

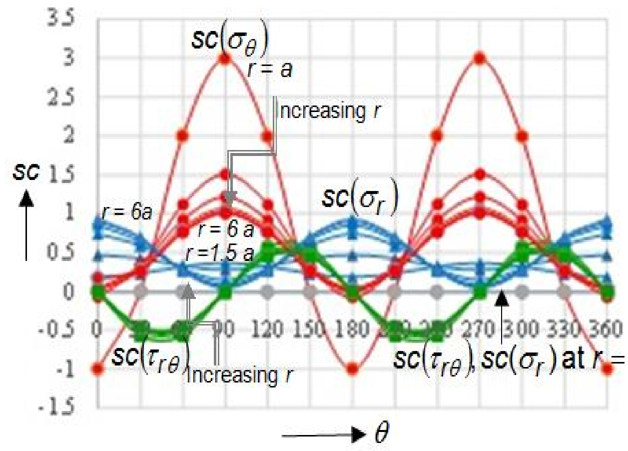
Using Tresca theory i.e. maximum shear stress should be less than the shear strength of the material

Case 3.1: $\left(\frac{3S-0}{2}\right) < \frac{YS}{2}$ or 3S < YS, Case 3.2: $2S < YS$, Case 3.3: $4S < YS$, Case 3.4: $4S < YS$

Using distortion energy theory i.e. von-Mises stress should be less than the yield strength of the material

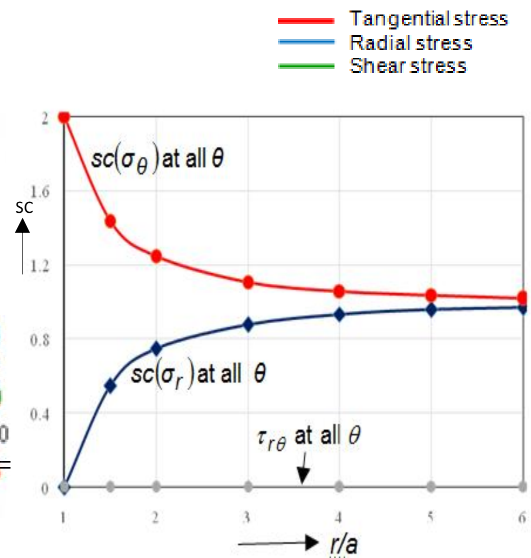
Case 3.1: $\sqrt{\frac{1}{2}\left\{(3S-0)^2+(3S-0)^2+(0-0)^2\right\}} < YS$ or 3S < YS, Case 3.2: $2S < YS$, Case 3.3: $4S < YS$, Case 3.4: $4S < YS$

b) When stress field at the critical points is in plastic state i.e. $(sc \times S) > YS$.



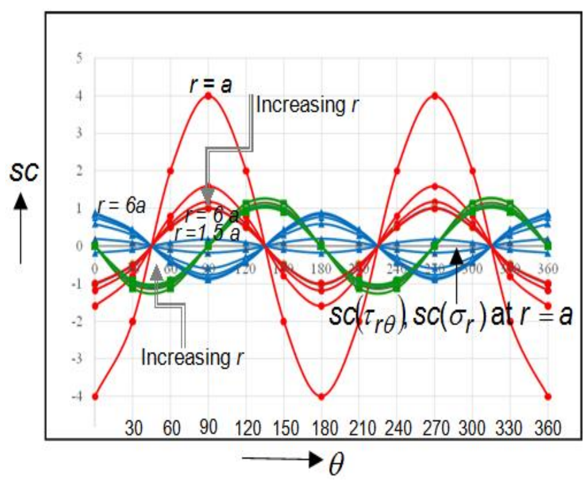
(A)

VARIATION OF STRESS CONCENTRATION *VS* ANGLE AND DISTANCE FROM HOLE IN CASE 3.1



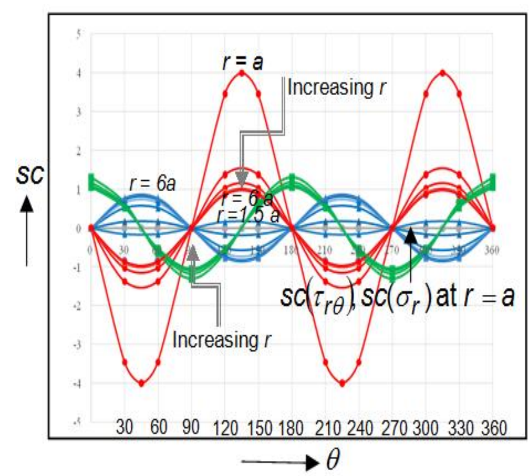
(B)

VARIATION OF STRESS CONCENTRATION *VS R/A* IN CASE 3.2



(C)

VARIATION OF STRESS CONCENTRATION *VS* ANGLE AND DISTANCE FROM HOLE IN CASE 3.3



(D)

VARIATION OF STRESS CONCENTRATION VS ANGLE AND DISTANCE FROM HOLE IN CASE 3.4

Fig. 2. Stress concentration plots

Plastic stress concentration factor that is defined by $sc_p = 1 + (sc - 1)\frac{E_s}{E}$ is used in such a case. sc_p is obtained in each of the cases with known sc values. sc_p is less than sc due to stress distribution caused by material yielding. Total strain, ε, corresponding to stress value $(sc_p \times S)$ on elastic-plastic curve of the material should be less than strain at fracture or $\varepsilon < \varepsilon_f$ for safe design.

4.1.2 Cyclic load

In cyclic load, sc is replaced by fatigue stress concentration factor or fatigue strength reduction factor, sc_f. Since only one principal stress exists at critical points in all the cases, fatigue at these points is of uni-axial type.

4.1.2.1 High cycle fatigue (HCF) i.e. when at critical point, σ_a < YS

sc_f in very HCF i.e. in endurance or fatigue limit cases involving infinite fatigue life is obtained from Neuber's equation, $sc_f = q(sc-1)+1$. Notch sensitivity index, *q*, depends upon the material grain size and contour radius of the discontinuity. Magnitude of sc_f in such cases is reported to be of the order of 2.3-2.5. In cases of HCF involving cyclic stress level above fatigue limit but less than the material yield strength, sc_f is found from experimentally obtained S-N curves (R = -1 where $R = \frac{\sigma_{min}}{\sigma_{max}}$) of specimens with and without the type of the discontinuity under consideration.

4.1.2.1.1 Ductile material

When mean stress of applied cycles is zero, i.e. R = -1, fatigue life, *N*, corresponding to applied cyclic stress amplitude, $\sigma_a \times sc_f$, is directly read from experimentally obtained S-N curve of the material at R = -1. When mean stress is non-zero, modified S-N curves are used. For example, Soderberg's criterion, $\frac{\sigma_a \times sc_f}{\sigma_N} + \frac{\sigma_m}{YS} = 1$ (sc_f is multiplied to only alternating component) provides the value of design cyclic stress amplitude, σ_N. Fatigue life, *N*, corresponding to σ_N is obtained from S-N curve at R = -1.

4.1.2.1.2 Brittle material

For zero mean stress, the procedure remains same as that for ductile material. For non-zero mean stress, Goodman's criterion for instance, $\frac{\sigma_a \times sc_f}{\sigma_N} + \frac{\sigma_m \times sc_f}{UTS} = 1$ provides the value of σ_N (sc_f is multiplied to both alternating and mean components). Fatigue life corresponding to σ_N is read from S-N curve at R = -1.

In both cases 4.1.2.1.1 and 4.1.2.1.2, modified Goodman's relations that are widely used in the industry give better results than stated criteria.

4.1.2.2 Low cycle fatigue (LCF) i.e. when at critical point, $\varepsilon_a > \frac{YS}{E}$

Discontinuities are experimentally found to have minimal effect on LCF behaviours. Many materials exhibit sc_f very near to unity at LCF lives of 10^3 to 10^4 cycles and less. Strain amplitudes at critical points may be assumed similar to far field or nominal strains. The conventional relationship governing $\varepsilon - N$ curve in LCF with use of nominal values, $\frac{\Delta\varepsilon}{2} = \frac{\sigma_f'}{E}(2N_f)^d + \varepsilon_f'(2N_f)^c$, provides the approximate number of reversals to failure, $2N_f$.

As detailed investigation, Neuber's rule for notches and Ramberg Osgood stress-strain equation can be used at critical points to obtain the parameters of interest in Cases 4.1.1.2 b) and 4.1.2.2 that involve plasticity at critical points.

5. SPECIAL CASES

Bi-axial loads can be un-identical in magnitude. Two cases where magnitude of loads can be dissimilar in longitudinal and transverse directions are discussed as under:-

5.1 Tensile-tensile

If $\sigma_x = S, \sigma_y = nS, \tau_{xy} = 0$ where $n > 1$

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \sin 2\theta = S\cos^2\theta + nS\sin^2\theta = \frac{S}{2}(1+n) + \frac{S}{2}(1-n)\cos 2\theta$$
$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - \tau_{xy} \sin 2\theta = S\sin^2\theta + nS\cos^2\theta = \frac{S}{2}(1+n) - \frac{S}{2}(1-n)\cos 2\theta$$
$$\tau_{r\theta} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}\cos 2\theta = -\frac{S}{2}(1-n)\sin 2\theta \tag{46}$$

Far field load can be split into axisymmetric and non-axisymmetric components as

a) $\sigma_r = \frac{S}{2}(1+n); \sigma_\theta = \frac{S}{2}(1+n); \tau_{r\theta} = 0$ (Axisymmetric)

b) $\sigma_r = \frac{S}{2}(1-n)\cos 2\theta; \sigma_\theta = -\frac{S}{2}(1-n)\cos 2\theta; \tau_{r\theta} = -\frac{S}{2}(1-n)\sin 2\theta$ (Non-axisymmetric)

For axisymmetric case, the solution is obtained on replacing S by $S(1+n)$ in Eq. (32) whereas for non-axisymmetric case, the solution is obtained on replacing S by $S(1-n)$ in Eq. (34). Final stress solution is obtained by superimposing individual solutions as follows:-

$$\sigma_r = \left[-\frac{a^2 S}{2r^2}(1+n) + \frac{S}{2}(1+n)\right] + \left[\frac{S}{2}(1-n) + \frac{6a^4 S}{4r^4}(1-n) - \frac{2a^2 S}{r^2}(1-n)\right]\cos 2\theta$$
$$\sigma_\theta = \left[\frac{a^2 S}{2r^2}(1+n) + \frac{S}{2}(1+n)\right] - \left[\frac{S}{2}(1-n) + \frac{6a^4 S}{4r^4}(1-n)\right]\cos 2\theta$$
$$\tau_{r\theta} = \left[-\frac{S}{2}(1-n) + \frac{6a^4 S}{4r^4}(1-n) - \frac{a^2 S}{r^2}(1-n)\right]\sin 2\theta \tag{47}$$

If $\sigma_x = nS, \sigma_y = S, \tau_{xy} = 0$ where $n > 1$

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \sin 2\theta = nS\cos^2\theta + S\sin^2\theta = \frac{S}{2}(1+n) + \frac{S}{2}(n-1)\cos 2\theta$$
$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - \tau_{xy} \sin 2\theta = nS\sin^2\theta + S\cos^2\theta = \frac{S}{2}(1+n) - \frac{S}{2}(n-1)\cos 2\theta$$
$$\tau_{r\theta} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}\cos 2\theta = -\frac{S}{2}(n-1)\sin 2\theta \tag{48}$$

Far field load can be split into axisymmetric and non-axisymmetric components as

a) $\sigma_r = \frac{S}{2}(1+n); \sigma_\theta = \frac{S}{2}(1+n); \tau_{r\theta} = 0$ (Axisymmetric)

b) $\sigma_r = \frac{S}{2}(n-1)\cos 2\theta; \sigma_\theta = -\frac{S}{2}(n-1)\cos 2\theta; \tau_{r\theta} = -\frac{S}{2}(n-1)\sin 2\theta$ (Non-axisymmetric)

For axisymmetric case, the solution is obtained on replacing S by $S(1+n)$ in Eq. (32) whereas for non-axisymmetric case, the solution is obtained on replacing S by $S(n-1)$ in Eq. (34). Final stress solution is obtained by superimposing individual solutions as follows:-

$$\sigma_r = \left[-\frac{a^2S}{2r^2}(1+n)+\frac{S}{2}(1+n)\right]+\left[\frac{S}{2}(n-1)+\frac{6a^4S}{4r^4}(n-1)-\frac{2a^2S}{r^2}(n-1)\right]\cos 2\theta$$
$$\sigma_\theta = \left[\frac{a^2S}{2r^2}(1+n)+\frac{S}{2}(1+n)\right]-\left[\frac{S}{2}(n-1)+\frac{6a^4S}{4r^4}(n-1)\right]\cos 2\theta$$
$$\tau_{r\theta} = \left[-\frac{S}{2}(n-1)+\frac{6a^4S}{4r^4}(n-1)-\frac{a^2S}{r^2}(n-1)\right]\sin 2\theta \quad (49)$$

5.2 Tensile-compressive

If $\sigma_x = S, \sigma_y = -nS, \tau_{xy} = 0$ where $n > 1$

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\sin 2\theta = S\cos^2\theta - nS\sin^2\theta = \frac{S}{2}(1-n)+\frac{S}{2}(1+n)\cos 2\theta$$
$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - \tau_{xy}\sin 2\theta = S\sin^2\theta - nS\cos^2\theta = \frac{S}{2}(1-n)-\frac{S}{2}(1+n)\cos 2\theta$$
$$\tau_{r\theta} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}\cos 2\theta = -\frac{S}{2}(1+n)\sin 2\theta \quad (50)$$

Far field load can be split into axisymmetric and non-axisymmetric components as

a) $\sigma_r = \frac{S}{2}(1-n); \sigma_\theta = \frac{S}{2}(1-n); \tau_{r\theta} = 0$ (Axisymmetric)

b) $\sigma_r = \frac{S}{2}(1+n)\cos 2\theta; \sigma_\theta = -\frac{S}{2}(1+n)\cos 2\theta; \tau_{r\theta} = -\frac{S}{2}(1+n)\sin 2\theta$ (Non-axisymmetric)

For axisymmetric case, the solution is obtained on replacing S by $S(1-n)$ in Eq. (32) whereas for non-axisymmetric case, the solution is obtained on replacing S by $S(1+n)$ in Eq. (34). Final stress solution is obtained by superimposing individual solutions as follows:-

$$\sigma_r = \left[-\frac{a^2S}{2r^2}(1-n)+\frac{S}{2}(1-n)\right]+\left[\frac{S}{2}(1+n)+\frac{6a^4S}{4r^4}(1+n)-\frac{2a^2S}{r^2}(1+n)\right]\cos 2\theta$$
$$\sigma_\theta = \left[\frac{a^2S}{2r^2}(1-n)+\frac{S}{2}(1-n)\right]-\left[\frac{S}{2}(1+n)+\frac{6a^4S}{4r^4}(1+n)\right]\cos 2\theta$$
$$\tau_{r\theta} = \left[-\frac{S}{2}(1+n)+\frac{6a^4S}{4r^4}(1+n)-\frac{a^2S}{r^2}(1+n)\right]\sin 2\theta \quad (51)$$

If $\sigma_x = -nS, \sigma_y = S, \tau_{xy} = 0$ where $n > 1$

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\sin 2\theta = -nS\cos^2\theta + S\sin^2\theta = \frac{S}{2}(1-n)-\frac{S}{2}(1+n)\cos 2\theta$$
$$\sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - \tau_{xy}\sin 2\theta = -nS\sin^2\theta + S\cos^2\theta = \frac{S}{2}(1-n)+\frac{S}{2}(1+n)\cos 2\theta$$
$$\tau_{r\theta} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}\cos 2\theta = \frac{S}{2}(1+n)\sin 2\theta \quad (52)$$

Far field load can be split into axisymmetric and non-axisymmetric components as

a) $\sigma_r = \frac{S}{2}(1-n); \sigma_\theta = \frac{S}{2}(1-n); \tau_{r\theta} = 0$ (Axisymmetric)

b) $\sigma_r = -\frac{S}{2}(1+n)\cos 2\theta; \sigma_\theta = \frac{S}{2}(1+n)\cos 2\theta; \tau_{r\theta} = \frac{S}{2}(1+n)\sin 2\theta$ (Non-axisymmetric)

For axisymmetric case, the solution is obtained on replacing S by $S(1-n)$ in Eq. (32) whereas for non-axisymmetric case, the solution is obtained on replacing S by $-S(1+n)$ in Eq. (34). Final stress solution is obtained by superimposing individual solutions as follows:-

$$\sigma_r = \left[-\frac{a^2S}{2r^2}(1-n)+\frac{S}{2}(1-n)\right]+\left[-\frac{S}{2}(1+n)-\frac{6a^4S}{4r^4}(1+n)+\frac{2a^2S}{r^2}(1+n)\right]\cos 2\theta$$
$$\sigma_\theta = \left[\frac{a^2S}{2r^2}(1-n)+\frac{S}{2}(1-n)\right]+\left[\frac{S}{2}(1+n)+\frac{6a^4S}{4r^4}(1+n)\right]\cos 2\theta$$
$$\tau_{r\theta} = \left[\frac{S}{2}(1+n)-\frac{6a^4S}{4r^4}(1+n)+\frac{a^2S}{r^2}(1+n)\right]\sin 2\theta \quad (53)$$

Equations of failure and fatigue life are written in similar manner as discussed previously in Section 4.1 for identical loads in longitudinal and transverse directions.

6. CONCLUSION

A comprehensive procedure to obtain closed form solutions of radial, tangential and shear stresses that develop in an infinite homogenous, isotropic and linear elastic plate with a circular hole under the action of various types of, uniform, axisymmetric and non-axisymmetric monotonic loads at infinity *viz.* i) Uni-axial tensile ii) Equal bi-axial (tensile-tensile) iii) Equal bi-axial (tensile-compressive) and iv) Pure shear stress is reviewed with the basic principles of elasticity. Fundamental bi-harmonic equation involving Airy's stress function is used in the analysis.

Stress near the hole is found to exceed the far field stress in each of the load cases due to the presence of discontinuity. Maximum stress in all the cases is normal and tangential in nature and is found to develop at the surface of the hole that gradually recedes and approaches far field stress state at locations far away from the hole. The nature and the magnitude of stress around the hole is influenced by the type of far field load and therefore differs in each load case. Maximum tangential stress is tensile and three times the applied stress in i), tensile and two times the applied stress in ii), four times the applied stress that assumes both tensile and compressive natures in iii) while the values in iv) are similar to that in iii) but locations of maximum stress differ. Monotonic and fatigue behaviour of plate made of linear elastic or elastic plastic material depends upon stress concentrations at critical points.

The present approach is applied to examine bi-axial loads of unidentical magnitudes as well. Since the paper investigates all types of basic loads that can possibly act over the structure, the cases involving inclined loads can be conveniently handled by resolving the loads into either of the investigated cases followed by superimposition of individual stress solutions to obtain the final solution. Similarly, solution of the cases involving both normal and shear loads can be found by superimposing the respective solutions.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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APPENDIX

Appendix A

Compatibility equation in 2D cartesian cordinate system is $\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}$.

Equilibrium equations, on ignoring body forces, are

$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$; $\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$ and can be written in consolidated form as

$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{2\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_y}{\partial y^2} = 0$ or $\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{1}{2}\left[\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right]$

a) *Plane stress condition,* (Small thickness, $\sigma_z = 0$)

Stress strain constitutive equations are $\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right]$; $\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right]$; $\gamma_{xy} = \frac{2(1+\nu)\tau_{xy}}{E}$

On substituting the constitutive equations in compatibiliy equation, we have

$$2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = \frac{\partial^2 \sigma_x}{\partial y^2} - \nu\frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - \nu\frac{\partial^2 \sigma_x}{\partial x^2} \qquad \text{(A.1)}$$

Using equilibrium equation in A.1, we obtain $\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial y^2} = 0$ or $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sigma_x + \sigma_y\right) = 0$

b) *Plane strain condition* (Large thickness, $\varepsilon_z = 0$)

The condition can be written in the form $\frac{1}{E}\left(\sigma_z - \nu\sigma_x - \nu\sigma_y\right) = 0$ or $\sigma_z = \nu\left(\sigma_x + \sigma_y\right)$

Stress strain constitutive equations are rewritten as

$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y - \nu\sigma_z\right]$; $\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x - \nu\sigma_z\right]$; $\gamma_{xy} = \frac{2(1+\nu)\tau_{xy}}{E}$

On using the value of σ_z in constitutive equations, we obtain

$\varepsilon_x = \frac{1}{E}\left[\sigma_x(1-\nu^2) - \sigma_y(\nu+\nu^2)\right]$; $\varepsilon_y = \frac{1}{E}\left[\sigma_y(1-\nu^2) - \sigma_x(\nu+\nu^2)\right]$; $\gamma_{xy} = \frac{2(1+\nu)\tau_{xy}}{E}$

On substituting the constitutive equations in compatibiliy equation, we have

$$2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = (1-\nu^2)\frac{\partial^2 \sigma_x}{\partial y^2} - (\nu+\nu^2)\frac{\partial^2 \sigma_y}{\partial y^2} + (1-\nu^2)\frac{\partial^2 \sigma_y}{\partial x^2} - (\nu+\nu^2)\frac{\partial^2 \sigma_x}{\partial x^2} \qquad \text{(A.2)}$$

Using equilibrium equation in A.2, we again obtain

$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial y^2} = 0$ or $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sigma_x + \sigma_y\right) = 0$

Appendix B

$\frac{\partial \sigma_x}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial y^2}\right) = \frac{\partial^3 \phi}{\partial x \partial y^2}$; $\frac{\partial \tau_{xy}}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial^2 \phi}{\partial x \partial y}\right) = -\frac{\partial^3 \phi}{\partial x \partial y^2}$. Hence $\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$

$\frac{\partial \tau_{yx}}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{\partial^2 \phi}{\partial x \partial y}\right) = -\frac{\partial^3 \phi}{\partial x^2 \partial y}$; $\frac{\partial \sigma_y}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 \phi}{\partial x^2}\right) = \frac{\partial^3 \phi}{\partial x^2 \partial y}$. Hence $\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$

Appendix C

Use of $x = r\cos\theta$ and $y = r\sin\theta$ as transformation from cartesian to polar coordinate system provides

$$r = \sqrt{x^2 + y^2} \text{ and } \frac{y}{x} = \tan\theta; \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}; \sec^2\theta \frac{\partial\theta}{\partial x} = -\frac{y}{x^2} \text{ or } \frac{\partial\theta}{\partial x} = -\frac{y}{x^2}\cos^2\theta = -\frac{y}{r^2}$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r}; \sec^2\theta \frac{\partial\theta}{\partial y} = \frac{1}{x} \text{ or } \frac{\partial\theta}{\partial y} = \frac{1}{x}\cos^2\theta = \frac{x}{r^2}$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial x} = \frac{x}{r}\frac{\partial\phi}{\partial r} - \frac{y}{r^2}\frac{\partial\phi}{\partial\theta}$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\frac{\partial\phi}{\partial r} - \frac{y}{r^2}\frac{\partial\phi}{\partial\theta}\right) = \frac{x}{r}\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial r}\right) + \frac{\partial\phi}{\partial r}\frac{\partial}{\partial x}\left(\frac{x}{r}\right) - \frac{y}{r^2}\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial\theta}\right) - \frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial x}\left(\frac{y}{r^2}\right)$$

The differential terms are simplified as follows : -

I. $\frac{x}{r}\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial r}\right) = \frac{x}{r}\left[\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial r}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{\partial r}\right)\frac{\partial\theta}{\partial x}\right] = \frac{x}{r}\left[\frac{\partial^2\phi}{\partial r^2}\frac{x}{r} - \frac{\partial^2\phi}{\partial r\partial\theta}\frac{y}{r^2}\right] = \frac{x^2}{r^2}\frac{\partial^2\phi}{\partial r^2} - \frac{xy}{r^3}\frac{\partial^2\phi}{\partial r\partial\theta}$

II. $\frac{\partial\phi}{\partial r}\frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{\partial\phi}{\partial r}\left[\frac{r - x\frac{\partial x}{\partial r}}{r^2}\right] = \frac{\partial\phi}{\partial r}\left[\frac{r^2 - x^2}{r^3}\right] = \frac{y^2}{r^3}\frac{\partial\phi}{\partial r}$

III. $\frac{y}{r^2}\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial\theta}\right) = \frac{y}{r^2}\left[\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial\theta}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{\partial\theta}\right)\frac{\partial\theta}{\partial x}\right] = \frac{y}{r^2}\left[\frac{\partial^2\phi}{\partial r\partial\theta}\frac{x}{r} - \frac{\partial^2\phi}{\partial\theta^2}\frac{y}{r^2}\right] = \frac{xy}{r^3}\frac{\partial^2\phi}{\partial r\partial\theta} - \frac{y^2}{r^4}\frac{\partial^2\phi}{\partial\theta^2}$

IV. $\frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial x}\left(\frac{y}{r^2}\right) = \frac{\partial\phi}{\partial\theta}\left[-\frac{2y}{r^3}\frac{\partial r}{\partial x}\right] = -\frac{2xy}{r^4}\frac{\partial\phi}{\partial\theta}$

Therefore $\frac{\partial^2\phi}{\partial x^2} = \frac{x^2}{r^2}\frac{\partial^2\phi}{\partial r^2} - \frac{2xy}{r^3}\frac{\partial^2\phi}{\partial r\partial\theta} + \frac{2xy}{r^4}\frac{\partial\phi}{\partial\theta} + \frac{y^2}{r^3}\frac{\partial\phi}{\partial r} + \frac{y^2}{r^4}\frac{\partial^2\phi}{\partial\theta^2}$ (C.1)

Likewise

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial y} = \frac{y}{r}\frac{\partial\phi}{\partial r} + \frac{x}{r^2}\frac{\partial\phi}{\partial\theta}$$

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{y}{r}\frac{\partial\phi}{\partial r} + \frac{x}{r^2}\frac{\partial\phi}{\partial\theta}\right) = \frac{y}{r}\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial r}\right) + \frac{\partial\phi}{\partial r}\frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{x}{r^2}\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial y}\left(\frac{x}{r^2}\right)$$

The differential terms are simplified as follows : -

I. $\frac{y}{r}\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial r}\right) = \frac{y}{r}\left[\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial r}\right)\frac{\partial r}{\partial y} + \frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{\partial r}\right)\frac{\partial\theta}{\partial y}\right] = \frac{y}{r}\left[\frac{\partial^2\phi}{\partial r^2}\frac{y}{r} + \frac{\partial^2\phi}{\partial r\partial\theta}\frac{x}{r^2}\right] = \frac{y^2}{r^2}\frac{\partial^2\phi}{\partial r^2} + \frac{xy}{r^3}\frac{\partial^2\phi}{\partial r\partial\theta}$

II. $\frac{\partial\phi}{\partial r}\frac{\partial}{\partial y}\left(\frac{y}{r}\right) = \frac{\partial\phi}{\partial r}\left[\frac{r - y\frac{\partial r}{\partial y}}{r^2}\right] = \frac{\partial\phi}{\partial r}\left[\frac{r^2 - y^2}{r^3}\right] = \frac{x^2}{r^3}\frac{\partial\phi}{\partial r}$

III. $\frac{x}{r^2}\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial\theta}\right) = \frac{x}{r^2}\left[\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial\theta}\right)\frac{\partial r}{\partial y} + \frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{\partial\theta}\right)\frac{\partial\theta}{\partial y}\right] = \frac{x}{r^2}\left[\frac{\partial^2\phi}{\partial r\partial\theta}\frac{y}{r} + \frac{\partial^2\phi}{\partial\theta^2}\frac{x}{r^2}\right] = \frac{xy}{r^3}\frac{\partial^2\phi}{\partial r\partial\theta} + \frac{x^2}{r^4}\frac{\partial^2\phi}{\partial\theta^2}$

IV. $\frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial y}\left(\frac{x}{r^2}\right) = \frac{\partial\phi}{\partial\theta}\left[-\frac{2x}{r^3}\frac{\partial r}{\partial y}\right] = -\frac{2xy}{r^4}\frac{\partial\phi}{\partial\theta}$

Therefore $\frac{\partial^2\phi}{\partial y^2}=\frac{y^2}{r^2}\frac{\partial^2\phi}{\partial r^2}+\frac{2xy}{r^3}\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{2xy}{r^4}\frac{\partial\phi}{\partial\theta}+\frac{x^2}{r^3}\frac{\partial\phi}{\partial r}+\frac{x^2}{r^4}\frac{\partial^2\phi}{\partial\theta^2}$ (C.2)

On adding (C.1) and (C.2), we have

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\left(\frac{x^2+y^2}{r^2}\right)\frac{\partial^2\phi}{\partial r^2}+\left(\frac{x^2+y^2}{r^3}\right)\frac{\partial\phi}{\partial r}+\left(\frac{x^2+y^2}{r^4}\right)\frac{\partial^2\phi}{\partial\theta^2}=\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

Appendix D

$$\frac{\partial\sigma_r}{\partial r}=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)=\frac{1}{r}\frac{\partial^2\phi}{\partial r^2}-\frac{1}{r^2}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^3\phi}{\partial r\partial\theta^2}-\frac{2}{r^3}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\frac{\partial\tau_{r\theta}}{r\partial\theta}=\frac{\partial}{r\partial\theta}\left[-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\right]=\frac{\partial}{r\partial\theta}\left(\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right)=\frac{1}{r^3}\frac{\partial^2\phi}{\partial\theta^2}-\frac{1}{r^2}\frac{\partial^3\phi}{\partial r\partial\theta^2}$$

$$\frac{1}{r}(\sigma_r-\sigma_\theta)=\frac{1}{r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}-\frac{\partial^2\phi}{\partial r^2}\right)=\frac{1}{r^2}\frac{\partial\phi}{\partial r}+\frac{1}{r^3}\frac{\partial^2\phi}{\partial\theta^2}-\frac{1}{r}\frac{\partial^2\phi}{\partial r^2}$$

Hence $\frac{\partial\sigma_r}{\partial r}+\frac{\partial\tau_{r\theta}}{r\partial\theta}+\frac{1}{r}(\sigma_r-\sigma_\theta)=0$

$$\frac{\partial\tau_{\theta r}}{\partial r}=\frac{\partial}{\partial r}\left[-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\right]=\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right)=-\frac{2}{r^3}\frac{\partial\phi}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial r\partial\theta}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial^3\phi}{\partial r^2\partial\theta}$$

$$\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}=\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial^2\phi}{\partial r^2}\right)=\frac{1}{r}\frac{\partial^3\phi}{\partial\theta\,\partial r^2}$$

$$\frac{2\tau_{\theta r}}{r}=\frac{2}{r}\left[-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\right]=\frac{2}{r}\left(\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right)=\frac{2}{r^3}\frac{\partial\phi}{\partial\theta}-\frac{2}{r^2}\frac{\partial^2\phi}{\partial r\partial\theta}$$

Hence $\frac{\partial\tau_{\theta r}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{2\tau_{\theta r}}{r}=0$

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