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New Modified Adomin Decomposition Method for Boundary Value Problems of Higher-order Ordinary Differential Equation

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

This study will present a new modified differential operator for solving third-order boundary value problems into higher-order ordinary differential equation. We found the differential operator for new three inverse operator which can be applied for solving equations at more than one type in different conditions. We put a detailed plan for five non-linear examples from a high-order, we get dynamic and quickly to the exact solution.

Keywords: Boundary value problems; adomain decomposition method; boundary conditions; higher-order nonlinear ODE.

1 Introduction

This paper studies Boundary Value Problems of the form:

$$y^{(n+2)} = f(x, y, y', ..., y^{(n+1)}), n \geq 1, \tag{1.1}$$

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with one of the following conditions

$$y(0) = r_0, y'(0) = r_1, ..., y^{(n-m)}(0) = r_n, y^{(n+m)}(0) = r_m, y^{(n-1)}(s) = k, \quad (1.2)$$

$$y(a) = b_0, y'(a) = b_1, ..., y^{(n)}(a) = b_n, y^{(n+m)}(0) = d, \quad (1.3)$$

$$y(c) = h_0, y'(c) = h_1, ..., y^{(n+1)}(c) = h_n. \quad (1.4)$$

Where f is a differential operator of linear or non-linear of order less than $(n+2)$. And $m = 0$ or $m = 1$, $a, b_0, b_1, ..., b_n, c, d, h_0, h_1, ..., h_n, r_0, r_1, ..., r_n, r_m$, s, k, are real finite constant.

The Boundary Value Problems (BVPs) consider chemical reactions, heat transfer, gas dynamics a nuclear physics as models for example BVPs [1]. There are numerous techniques solutions for BVPs is considered as a decisive dot in scientific account [2]. For obtaining solutions, a lot trying have been made by investigators to resolve these models by developing new techniques. According to my reading, we found only slight studies with regard to numerical solutions of higher-order BVPs in literature [3-5,6].

The Adomain decomposition method (ADM) [7,8,9], has been studied by many scientists for solving differential and integral problems in many scientific and physical applications. It resolve the solution into the series which converges quickly. In this paper, a new modified of the Adomain decomposition method (MADM) is proposed to find a differential operator as well as its inverse operator in order to solve BVP. This paper aims to employ the new MADM which can be used for solution of higher-order boundary value problem under various kinds of different conditions to solve an equation at more than one condition. The present study analyzing method. A lot of numerical techniques commentary are illustrate in the following.

2 Analysis of the Method

To study the equation (1.1), we would suggest the new differential operator,

$$L(.) = \frac{d^m}{dx^m} x^{-1} \frac{d^{2-m}}{dx^{2-m}} x^{3-m} \frac{d}{dx} x^{m-2} \frac{d^{n-1}}{dx^{n-1}}(.), \quad (2.1)$$

now, can be written the equation (1.1) as,

$$Ly = f(x, y, y', y'', ..., y^{(n+1)}), \quad (2.2)$$

under one of the conditions (1.2), (1.3) and (1.4), for three inverse operators L^{-1} is given, respectively as

$$L^{-1} = \underbrace{\int_0^x \int_0^x \int_0^x ... \int_0^x}_{(n-1)} x^{2-m} \int_b^x x^{m-3} \underbrace{\int_0^x \int_0^x ... \int_0^x}_{(2-m)} x \underbrace{\int_0^x \int_0^x ... \int_0^x}_{(m)} (.) \underbrace{dxdx...dx.}_{(n+2)-times}$$

$$L^{-1} = \underbrace{\int_a^x \int_a^x \int_a^x ... \int_a^x}_{(n-1)} x^{2-m} \int_a^x x^{m-3} \underbrace{\int_a^x \int_0^x ... \int_0^x}_{(2-m)} x \underbrace{\int_0^x \int_0^x ... \int_0^x}_{(m)} (.) \underbrace{dxdx...dx.}_{(n+2)-times}$$

$$L^{-1} = \underbrace{\int_c^x \int_c^x \int_c^x ... \int_c^x}_{(n-1)} x^{2-m} \int_c^x x^{m-3} \underbrace{\int_c^x \int_c^x ... \int_c^x}_{(2-m)} x \underbrace{\int_c^x \int_c^x ... \int_c^x}_{(m)} (.) \underbrace{dxdx...dx.}_{(n+2)-times}$$

By applying L^{-1} on (2.2), we obtain

$$y(x) = \phi(x) + L^{-1} f(x, y, y', ..., y^{(n+1)}), \quad (2.3)$$

where $\phi(x)$ represent the terms arising from using auxiliary conditions. The Adomain decomposition method represent the solution $y(x)$ and the non-linear function $f(x, y, y', y'', ..., y^{(n+1)})$ by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{2.4}$$

and

$$f(x, y, y', y'', ..., y^{(n+1)}) = \sum_{n=0}^{\infty} A_n, \tag{2.5}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently by algorithm [10,11,12].

A_n are the Adomain polynomials, which are obtain formula the following

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i y_i\right)\right]_{\lambda=0}, n = 0, 1, 2, ...,$$

which gives

$$\begin{aligned} &A_0 = F(y_0),\\ &A_1 = y_1 F'(y_0),\\ &A_2 = y_2 F'(y_0) + y_1^2 \tfrac{1}{2} F''(y_0),\\ &A_3 = y_3 F'(y_0) + y_1 y_2 F''(y_0) + y_1^3 \tfrac{1}{3!} F'''(y_0), \end{aligned} \tag{2.6}$$

$$...$$

Substituting eq.(2.4) and eq.(2.5) into eq.(2.3), we get

$$\sum_{n=0}^{\infty} y_n(x) = \phi(x) + L^{-1}\sum_{n=0}^{\infty} A_n, \tag{2.7}$$

we get the components y_n can be specified as

$$y_0 = \phi(x),$$

$$y_{n+1} = L^{-1}A_n, n \geq 0,$$

which gives

$$\begin{aligned} &y_0 = \phi(x),\\ &y_1 = L^{-1}A_0,\\ &y_2 = L^{-1}A_1,\\ &y_3 = L^{-1}A_2, \end{aligned} \tag{2.8}$$

$$...$$

From (2.6) and (2.8), we find the components $y_n(x)$, and hence the series solution of $y(x)$ in (2.7) can be directly obtained. For numerical aim, the $n-$ term approximate

$$\Psi(x) = \sum_{k=0}^{n-1} y_k$$

can be used to approximate the exact solution. The approach above can be support by testing it on a variety of several linear and nonlinear BVP.

3 Application of MADM

In this part, when n=1,2,4, in a differential operator (2.1). We apply the proposed algorithm on two third order non-linear boundary value problems at m=0& m=1, two fourth order non-linear boundary value problems at m=0 & m=1 and one sixth order non-linear boundary value problem at m=0& m=1 and in every one case three boundary conditions.

3.1 Example

The first case, when n=1 and m=0, we give example non-linear equation of third order:

$$y'''(x) = y^2 - y - x^2(x^2 - 1), \tag{3.1.1}$$

under one of the following conditions

$$y(0) = 1, y'(0) = 0, y(1) = 0,$$

$$y(1) = 0, y'(1) = -2, y'(0) = 0,$$

$$y(\frac{1}{2}) = \frac{3}{4}, y'(\frac{1}{2}) = -1, y''(\frac{1}{2}) = -2.$$

The exact solution is $y(x) = 1 - x^2$.

Can be written eq. (3.1.1), as

$$Ly = y^2 - y - x^2(x^2 - 1), \tag{3.1.2}$$

from an operator (2.1), give

$$L(.) = x^{-1}\frac{d^2}{dx^2}x^3\frac{d}{dx}x^{-2}(.),$$

for three inverse operators under one of the following conditions, respectively

$$L^{-1}(.) = x^2\int_1^x x^{-3}\int_0^x\int_0^x x(.)\ dxdxdx.$$

$$L^{-1}(.) = x^2\int_1^x x^{-3}\int_1^x\int_0^x x(.)\ dxdxdx.$$

$$L^{-1}(.) = x^2\int_{\frac{1}{2}}^x x^{-3}\int_{\frac{1}{2}}^x\int_{\frac{1}{2}}^x x(.)\ dxdxdx.$$

Applying L^{-1} to both sides of (3.1.2) and using the boundary conditions, we obtain respectively

$$y(x) = 1 - x^2 + L^{-1}y^2 - L^{-1}y - L^{-1}x^2(x^2 - 1),$$

$$y(x) = 1 - x^2 + L^{-1}y^2 - L^{-1}y - L^{-1}x^2(x^2 - 1),$$

$$y(x) = 1 - x^2 + L^{-1}y^2 - L^{-1}y - L^{-1}x^2(x^2 - 1),$$

employing ADM for $y^2(x)$, as yield for every one above

$$\sum_{n=0}^{\infty} y_n(x) = 1 - x^2 - L^{-1}x^2(x^2 - 1) - L^{-1}y_n + L^{-1}\sum_{n=0}^{\infty} A_n, n \geq 0,$$

the components for $y_n(x)$, introduces the recursive relation, respectively

$$y_0 = 1. - 1.0119\,x^2 + 0.0166667\,x^5 - 0.0047619\,x^7,$$

$$y_0 = 1.0131 - 1.025\,x^2 + 0.0166667\,x^5 - 0.0047619\,x^7,$$

$$y_0 = 0.997433 + 0.0130208\,x - 1.01771\,x^2 + 0.0166667\,x^5 - 0.0047619\,x^7,$$

$$y_{n+1} = -L^{-1}y_n + L^{-1}A_n, n \geq 0,$$

applying Adomain polynomial A_n, for the non-linear term y^2, when for n=0,1, gives

$$A_0 = y_0^2,$$

$$A_1 = 2y_0y_1,$$

Then, we can proceed to compute the first few components respectively, as follows

$$y_1 = 0.0119856\,x^2 - 0.0168651\,x^5 + 0.00487596\,x^7 + ... + 0.0000534612\,x^{10},$$

$$y_2 = -0.0000810072\,x^2 + 0.000199759\,x^5 + ... + 0.0000547322\,x^{10},$$

$$y_1 = -0.0261226 + 0.128954\,x - 0.236643\,x^2 + 0.1733\,x^3 + ... + 5.80154\,10^{-10}\,x^{20},$$

$$y_2 = -0.000201743 - 0.0044678\,x^3 + 0.0055138\,x^4 + ... + 6.99862\,10^{-10}\,x^{20},$$

$$y_1 = -0.00256696 + 0.0130208\,x - 0.0177083\,x^2 + 0.0166667\,x^5 - 0.0047619\,x^7,$$

The first terms, the approximate is following, respectively

$$y(x) = y_0 + y_1 + y_2 = 1. - 1.\,x^2 + 1.34678\,10^{-6}\,x^5 + ... + 5.35508\,10^{-14}\,x^{25},$$

$$y(x) = y_0 + y_1 + y_2 = 0.986771 + 0.128954\,x - 1.26164\,x^2 + .. + 1.19708\,10^{-10}\,x^{20},$$

$$y(x) = y_0 + y_1 = 0.994866 + 0.0260417\,x - 1.03542\,x^2 + 0.0333333\,x^5 - 0.00952381\,x^7,$$

Table 3.1. The comparison between exact solution and MADM under three conditions

x	Exact solution	MADN at the first condition	Absolute Error	MADM at the second condition	Absolute Error	MADM at the third condition	Absolute Error
0.1	0.99	0.99	0.00	0.987219	0.002781	0.9871160	0.002884
0.2	0.96	0.96	0.00	0.963443	0.003443	0.9586680	0.001332
0.3	0.91	0.91	0.00	0.916419	0.006419	0.9095700	0.000430
0.4	0.84	0.84	0.00	0.847052	0.007052	0.8399420	0.000058
0.5	0.75	0.75	0.00	0.756144	0.006144	0.7500000	0.000000
0.6	0.64	0.64	0.00	0.644379	0.004379	0.6400066	0.0000066
0.7	0.51	0.51	0.00	0.512316	0.002316	0.5105590	0.000559
0.8	0.36	0.36	0.00	0.360398	0.000398	0.3619580	0.001958
0.9	0.19	0.19	0.00	0.188956	0.001044	0.1947440	0.004744
1.0	0.00	0.00	0.00	-0.001791	0.001791	0.093006	0.093006

We see from Table 3.1, the error less than possible, and the first condition is the exact solution.

We notice in the Fig. 3.1 the convergence between the exact solution and the approximate solutions under the boundary conditions.

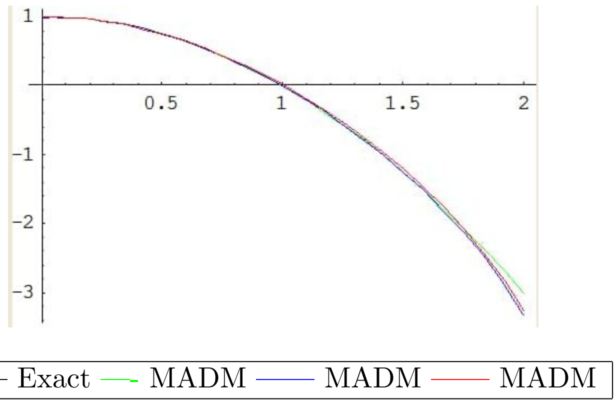

Fig. 3.1. Comparison between exact solution and MADM under three conditions, respectively

3.2 Example

In this case and at the same time m=1, below example non-linear of third order:

$$y'''(x) = 60x^2 + x^{10}y - y^3, \tag{3.2.1}$$

under one of the following conditions

$$y(0) = 0, y''(0) = 0, y(1) = 1,$$

$$y(\frac{1}{2}) = \frac{1}{32}, y'(\frac{1}{2}) = \frac{5}{16}, y''(0) = 0,$$

$$y(\frac{1}{2}) = \frac{1}{32}, y'(\frac{1}{2}) = \frac{5}{16}, y''(\frac{1}{2}) = \frac{5}{2}.$$

The exact solution is $y(x) = x^5$. Can be written eq. (3.2.1), as

$$Ly = 60x^2 + x^{10}y - y^3, \tag{3.2.2}$$

from an operator (2.1), we get

$$L(.) = \frac{d}{dx}x^{-1}\frac{d}{dx}x^2\frac{d}{dx}x^{-1}(.),$$

for three inverse operators under one of the following conditions respectively,

$$L^{-1}(.) = x\int_1^x x^{-2}\int_0^x x\int_0^x (.)\ dxdxdx.$$

$$L^{-1}(.) = x\int_{\frac{1}{2}}^x x^{-2}\int_{\frac{1}{2}}^x x\int_0^x (.)\ dxdxdx.$$

$$L^{-1}(.) = x\int_{\frac{1}{2}}^x x^{-2}\int_{\frac{1}{2}}^x x\int_{\frac{1}{2}}^x (.)\ dxdxdx.$$

Applying L^{-1}, to both sides of (3.2.2) and using the boundary conditions, give respectively

$$y(x) = x + L^{-1}60x^2 + L^{-1}x^{10}y - L^{-1}y^3,$$

$$y(x) = \frac{-1}{8} + \frac{5}{16}x + L^{-1}60x^2 + L^{-1}x^{10}y - L^{-1}y^3,$$
$$y(x) = \frac{3}{16} - \frac{15}{16}x + \frac{5}{4}x^2 + L^{-1}60x^2 + L^{-1}x^{10}y - L^{-1}y^3,$$

employing ADM for $y^3(x)$, as yield for every one above

$$\sum_{n=0}^{\infty} y_n(x) = x + L^{-1}60x^2 + L^{-1}x^{10}y_n - L^{-1}\sum_{n=0}^{\infty} A_n, n \geq 0,$$
$$\sum_{n=0}^{\infty} y_n(x) = \frac{-1}{8} + \frac{5}{16}x + L^{-1}60x^2 + L^{-1}x^{10}y_n - L^{-1}\sum_{n=0}^{\infty} A_n, n \geq 0,$$
$$\sum_{n=0}^{\infty} y_n(x) = \frac{3}{16} - \frac{15}{16}x + \frac{5}{4}x^2 + L^{-1}60x^2 + L^{-1}x^{10}y_n - L^{-1}\sum_{n=0}^{\infty} A_n, n \geq 0,$$

the components for $y_n(x)$ introduces the recursive relation, respectively

$$y_0 = x^5,$$
$$y_0 = x^5,$$
$$y_0 = x^5,$$
$$y_{n+1} = L^{-1}x^{10}y_n - L^{-1}A_n, n \geq 0,$$

the first components respectively, as follows

$$y_1 = 0,$$
$$y_2 = 0,$$

$$y_1 = 0,$$
$$y_2 = 0,$$

$$y_1 = 0,$$
$$y_2 = 0,$$

This means that the solution in a series from is following

$$y(x) = y_0 + y_1 + y_2 =$$
$$y(x) = x^5.$$

Plainly, the previous example, we have the exact solution. Thus the good method and its effectiveness.

3.3 Example

The second case, we give example for non-linear of fourth order, at n=2,m=0

$$y^{(4)} = (y')^2 - yy'' - 4x^2 + e^x(1 + x^2 - 4x), \quad (3.3.1)$$

under one of the following conditions

$$y(0) = 1, y'(0) = 1, y''(0) = 3, y'(\frac{1}{2}) = 3.72,$$
$$y(1) = 3.72, y'(1) = 4.72, y'''(1) = 4.72, y''(0) = 3,$$
$$y(\frac{1}{2}) = 1.9, y'(\frac{1}{2}) = 2.65, y''(\frac{1}{2}) = 3.65, y'''(\frac{1}{2}) = 1.65.$$

The exact solution is $y(x) = e^x + x^2$. Can be written eq.(3.3.1), gives

$$Ly = (y')^2 - yy'' - 4x^2 + e^x(1 + x^2 - 4x), \tag{3.3.2}$$

from an operator (2.1), we get

$$L(.) = x^{-1}\frac{d^2}{dx^2}x^3\frac{d}{dx}x^{-2}\frac{d}{dx}(.),$$

for three inverse operators under one of the following conditions, respectively

$$L^{-1}(.) = \int_0^x x^2 \int_{\frac{1}{2}}^x x^{-3} \int_0^x \int_0^x x(.)\ dxdxdx,$$

$$L^{-1}(.) = \int_1^x x^2 \int_1^x x^{-3} \int_1^x \int_0^x x(.)\ dxdxdx,$$

$$L^{-1}(.) = \int_{\frac{1}{2}}^x x^2 \int_{\frac{1}{2}}^x x^{-3} \int_{\frac{1}{2}}^x \int_{\frac{1}{2}}^x x(.)\ dxdxdx.$$

Applying L^{-1}, to both sides of (3.3.2) and using the boundary conditions, we give respectively

$$y(x) = 1 + x + 1.5x^2 + 0.853x^3 + L^{-1}e^X(1 + x^2 - 4x) - L^{-1}4x^2 + L^{-1}(y')^2 - L^{-1}yy'',$$

$$y(x) = 1.07 + 0.86x + 1.5x^2 + 0.29x^3 + L^{-1}e^X(1 + x^2 - 4x) - L^{-1}4x^2 + L^{-1}(y')^2 - L^{-1}yy'',$$

$$y(x) = 0.9998 + 1.03x + 1.0415x^2 + 0.273x^3 + L^{-1}e^X(1 + x^2 - 4x) - L^{-1}4x^2 + L^{-1}(y')^2 - L^{-1}yy'',$$

employing ADM for $(y')^2 - yy''$, as yield for all above

$$\sum_{n=0}^{\infty} y_n(x) = 1 + x + 1.5x^2 + 0.853x^3 + L^{-1}e^X(1 + x^2 - 4x) - L^{-1}4x^2 + L^{-1}\sum_{n=0}^{\infty} A_n - L^{-1}y_n y_n'', n \geq 0,$$

$$\sum_{n=0}^{\infty} y_n(x) = 1.07 + 0.86x + 1.5x^2 + 0.29x^3 + L^{-1}e^X(1 + x^2 - 4x) - L^{-1}4x^2 + L^{-1}\sum_{n=0}^{\infty} A_n - L^{-1}y_n y_n'', n \geq 0,$$

$$\sum_{n=0}^{\infty} y_n(x) = 0.9998 + 1.03x + 1.415x^2 + 0.273x^3 + L^{-1}e^X(1 + x^2 - 4x) - L^{-1}4x^2 + L^{-1}\sum_{n=0}^{\infty} A_n - L^{-1}y_n y_n'', n \geq 0,$$

the components for $y_n(x)$ introduces the recursive relation, respectively

$$y_0 = 1 + x + 1.5x^2 + 0.853x^3 + L^{-1}e^X(1 + x^2 - 4x) - L^{-1}4x^2,$$

$$y_0 = 1.07 + 0.86x + 1.5x^2 + 0.29x^3 + L^{-1}e^X(1 + x^2 - 4x) - L^{-1}4x^2,$$

$$y_0 = 0.9998 + 1.03x + 1.415x^2 + 0.273x^3 + L^{-1}e^X(1 + x^2 - 4x) - L^{-1}4x^2,$$

$$y_{n+1} = L^{-1}A_n - L^{-1}y_n y_n'', n \geq 0,$$

the first three terms respectively, as follows:

$$y_0 = 1 + x + 1.5\,x^2 + 0.840303\,x^3 + 0.0416667\,x^4 - 0.025\,x^5 + ... + 5.85008\,10^{-10}\,x^{14},$$

$$y_1 = 0.166028\,x^3 - 0.210076\,x^4 - 0.0503485\,x^5 - 0.0196187\,x^6 + ... + 4.25391\,10^{-17}\,x^{25},$$

$$y_2 = -0.000676185\,x^3 + 0.00276714\,x^6 + 0.00136013\,x^7 + ... + 2.34098\,10^{-35}\,x^{50},$$

$$y_0 = 1.19399 + 0.640618\,x + 1.5\,x^2 + 0.387846\,x^3 + ... + 1.9290110^{-6}x^{10},$$

$$y_1 = -0.177877 + 0.292906\,x - 0.0631219\,x^3 - 0.0986711\,x^4 + ... + 3.44211\,10^{-17}\,x^{25},$$

$$y_2 = -0.0302707 + 0.0613275\,x - 0.0558587\,x^3 + 0.0128297\,x^4 + ... + 4.5765\,10^{-35}\,x^{50},$$

$$y_0 = 0.996293 + 1.05407\,x + 1.36146\,x^2 + 0.2996\,x^3 + 0.0416667\,x^4 + ... + 5.85008\,10^{-10}\,x^{14},$$

$$y_1 = 0.00625454 - 0.0440354\,x + 0.105108\,x^2 - 0.0790648\,x^3 + ... + 1.21385\,10^{-20}\,x^{28},$$

$$y_2 = -0.0000184692 + 0.000235623\,x - 0.00123993\,x^2 + ... + 1.40914\,10^{-12}\,x^{24},$$

In this method, any assistance can be obvious calculated at any order. If we solve for the first term, the approximate is following, respectively

$$y(x) = y_0 + y_1 + y_2 =$$

$$y(x) = 1. + x + 1.5\,x^2 + 1.00566\,x^3 - 0.168409\,x^4 + ... + 2.34098\,10^{-35}\,x^{50},$$

$$y(x) = 0.985838 + 0.994851\,x + 1.5\,x^2 + 0.268866\,x^3 + ... + 4.5765\,10^{-35}\,x^{50},$$

$$y(x) = 1.00253 + 1.01027\,x + 1.46533\,x^2 + 0.223916\,x^3 + ... + 1.21385\,10^{-20}\,x^{28},$$

Table 3.3. The comparison between exact solution and MADM under three conditions

x	Exact solution	MADN at the first condition	Absolute Error	MADM at the second condition	Absolute Error	MADM at the third condition	Absolute Error
0.0	1.00000	1.00000	0.00000	0.98600	0.01400	1.00250	0.00250
0.1	1.11517	1.11599	0.00082	1.00590	0.10927	1.11843	0.00326
0.2	1.26140	1.26775	0.00635	1.24689	0.01451	1.26500	0.00360
0.3	1.43986	1.46058	0.02072	1.42619	0.01367	1.44361	0.00375
0.4	1.65182	1.69913	0.04731	1.63987	0.01195	1.65568	0.00386
0.5	1.89872	1.98727	0.08855	1.88920	0.00952	1.90267	0.00395
0.6	2.18212	2.32786	0.14574	2.17546	0.00666	2.18618	0.00406
0.7	2.50375	2.72259	0.21884	2.49997	0.00378	2.50793	0.00418
0.8	2.86554	3.171169	0.3056	2.86422	0.00132	2.86998	0.00444
0.9	3.26960	3.67369	0.40409	3.27008	0.00048	3.27484	0.00524
1.0	3.71828	4.22504	0.50676	3.72000	0.00172	3.72579	0.00751

We notice in the Fig. 3.3 the convergence between the exact solution and the approximate solutions under the boundary conditions. And we have access to the solution exact.

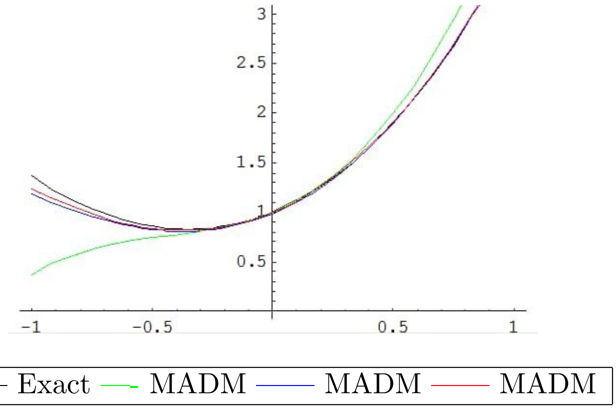

Fig. 3.3. Comparison between exact solution $y(x) = e^x + x^2$, and MADM at three conditions $\sum_{n=0}^{2} y_n(x)$, respectively

3.4 Example

In this case and at the same time m=1, we give example non-linear of fourth-order [13]:

$$y''''(x) = e^{-x}y^2(x), \tag{3.4.1}$$

under one of the following conditions

$$y(0) = 1, y'(0) = 1, y'''(0) = 1, y'(1) = 2.72,$$
$$y(\frac{1}{2}) = 1.65, y'(\frac{1}{2}) = 1.65, y''(\frac{1}{2}) = 1.65, y'''(0) = 1.$$
$$y(1) = 2.7, y'(1) = 2.7, y''(1) = 2.7, y'''(1) = 2.7,$$

The exact solution is $y(x) = e^x$. Can be written eq.(3.4.1), as

$$Ly = e^{-x}y^2(x), \tag{3.4.2}$$

from an operator (2.1), when m=1,n=2, we get

$$L(.) = \frac{d}{dx}x^{-1}\frac{d}{dx}x^2\frac{d}{dx}x^{-1}\frac{d}{dx}(.),$$

for three inverse operators under one of the following conditions respectively,

$$L^{-1}(.) = \int_0^x x\int_1^x x^{-2}\int_0^x x\int_0^x (.)\ dxdxdxdx.$$
$$L^{-1}(.) = \int_{\frac{1}{2}}^x x\int_{\frac{1}{2}}^x x^{-2}\int_{\frac{1}{2}}^x x\int_0^x (.)\ dxdxdxdx.$$
$$L^{-1}(.) = \int_1^x x\int_1^x x^{-2}\int_1^x x\int_1^x (.)\ dxdxdxdx.$$

Applying L^{-1}, to both sides of (3.4.2) and using the boundary conditions, we give respectively

$$y(x) = 1 + x + 0.61x^2 + 0.167x^3 + e^{-x}y^2(x),$$

$$y(x) = 0.997 + 0.95x + 0.575x^2 + 0.167x^3 + L^{-1}e^{-x}y^2(x),$$

$$y(x) = 0.907 + 1.36x + 0.453x^3 + L^{-1}e^{-x}y^2(x),$$

employing ADM for $y^2(x)$, as yield for every one above

$$\sum_{n=0}^{\infty} y_n(x) = 1 + x + 0.61x^2 + 0.167x^3 + L^{-1}\sum_{n=0}^{\infty} e^{-x}A_n, n \geq 0,$$

$$\sum_{n=0}^{\infty} y_n(x) = 0.997 + 0.95x + 0.575x^2 + 0.167x^3 + L^{-1}\sum_{n=0}^{\infty} e^{-x}A_n, n \geq 0,$$

$$\sum_{n=0}^{\infty} y_n(x) = 0.907 + 1.36x + 0.453x^3 + L^{-1}\sum_{n=0}^{\infty} e^{-x}A_n, n \geq 0,$$

the components for $y_n(x)$ introduces the recursive relation, respectively

$$y_0 = 1 + x + 0.61x^2 + 0.167x^3,$$

$$y_0 = 0.997 + 0.95x + 0.575x^2 + 0.167x^3,$$

$$y_0 = 0.907 + 1.36x + 0.453x^3,$$

$$y_{n+1} = L^{-1}e^{-x}A_n, n \geq 0,$$

the first components respectively, as follows

$$y_1 = -0.110759\, x^2 + 0.0416667\, x^4 + 0.00833333\, x^5 + 0.002\, x^6 + ... + 6.58308\, 10^{-14}\, x^{20},$$

$$y_2 = 0.00164057\, x^2 - 0.000615327\, x^6 + 0.000035099\, x^8 + ... + 1.36814\, 10^{-13}\, x^{20},$$

$$y_1 = -0.00947569 + 0.0499038\, x - 0.0733157\, x^2 + 0.041417\, x^4 + ... + 6.60944\, 10^{-14}\, x^{20},$$

$$y_2 = 0.000051099 - 0.000293793\, x + 0.000500661\, x^2 + ... + 3.97136\, 10^{-13}\, x^{20},$$

$$y_1 = 0.0939159 - 0.358699\, x + 0.497474\, x^2 - 0.280518\, x^3 + ... + 4.86327\, 10^{-13}\, x^{20},$$

$$y_2 = 0.000120243 - 0.000944067\, x + 0.00321515\, x^2 + ... + 7.94958\, 10^{-10}\, x^{20},$$

Thus, respectively

$$y(x) = y_0 + y_1 + y_2 =$$

$$y(x) = 1 + x + 0.500882\, x^2 + \frac{x^3}{6} + 0.0416667\, x^4 + 0.00833333\, x^5 + 0.00138467\, x^6 + 0.000198413\, x^7 + 0.0000174998\, x^8 + 6.81227\, 10^{-6}\, x^9 + 1.13245\, 10^{-6}\, x^{10} - 6.49119\, 10^{-7}\, x^{11} + 7.60813\, 10^{-8}\, x^{12} + 6.6514\, 10^{-9}\, x^{13} + ... + 3.81499\, 10^{-27}\, x^{37},$$

$$y(x) = 0.987575 + 0.99961\, x + 0.502185\, x^2 + 0.167\, x^3 + 0.0406298\, x^4 + 0.00833908\, x^5 + 0.00138481\, x^6 + 0.000213127\, x^7 + 0.0000161673\, x^8 + 3.84595\, 10^{-6}\, x^9 + 1.56133\, 10^{-6}\, x^{10} - 6.48035\, 10^{-7}\, x^{11} + 5.33129\, 10^{-8}\, x^{12} + ... + 3.83793\, 10^{-27}\, x^{37},$$

$$y(x) = 1.00104 + 1.00036\, x + 0.50069\, x^2 + 0.166327\, x^3 + 0.0413755\, x^4 + 0.00898998\, x^5 + 0.000558478\, x^6 + 0.00100613\, x^7 - 0.000571726\, x^8 + 0.0003373\, x^9 - 0.000141281\, x^{10} + 0.0000445961\, x^{11} - 0.0000101886\, x^{12} + 1.61948\, 10^{-6}\, x^{13} + ... + 7.66024\, 10^{-26}\, x^{37},$$

Table 3.4. The comparison between exact solution and MADM under three conditions

x	Exact solution	MADN at the first condition	Absolute Error	MADM at the second condition	Absolute Error	MADM at the third condition	Absolute Error
0.0	1.00000	1.00000	0.0000	0.987575	0.012425	1.00104	0.00104
0.1	1.10517	1.10518	0.00001	1.09273	0.01244	1.10625	0.00108
0.2	1.22140	1.22144	0.00004	1.20899	0.01241	1.22253	0.00113
0.3	1.34986	1.34994	0.00008	1.33751	0.01235	1.35105	0.00119
0.4	1.49182	1.49197	0.00015	1.47959	0.01223	1.49309	0.00127
0.5	1.64872	1.64894	0.00022	1.63662	0.01210	1.65006	0.00134
0.6	1.82212	1.82244	0.00032	1.81018	0.01194	1.82353	0.00141
0.7	2.01375	2.01418	0.00043	2.00199	0.01176	2.01525	0.0015
0.8	2.22554	2.22610	0.00056	2.21395	0.01159	2.22711	0.00157
0.9	2.45960	2.46031	0.00071	2.44817	0.01143	2.46125	0.00165
1.0	2.71828	2.71916	0.00088	2.70696	0.01132	2.72000	0.00172

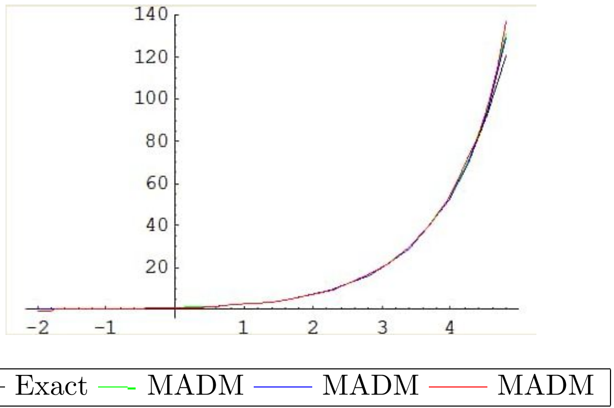

Fig. 3.4. Comparison between exact solution $y(x) = e^x$, and MADM at three conditions $\sum_{n=0}^{2} y_n(x)$, respectively

We notice in the figure above the convergence between the exact solution and the approximate solutions under the boundary conditions. And we have access to the solution exact.

3.5 Example

In the last case, we will give example non-linear of sixth-order [13], at m=0,1,

$$\frac{d^6y}{dx^6} = e^{-x}y^2(x), \tag{3.5.1}$$

under one of the following condition

$$y(0) = 1, y'(0) = 1, y''(0) = 1, y'''(0) = 1, y''''(0) = 1, y'''(1) = e,$$
$$y(1) = e, y'(1) = e, y''(1) = e, y'''(1) = e, y''''(1) = e, y''''(0) = 1,$$

$$y(\frac{1}{2}) = e^{\frac{1}{2}}, y'(\frac{1}{2}) = e^{\frac{1}{2}}, y''(\frac{1}{2}) = e^{\frac{1}{2}}, y'''(\frac{1}{2}) = e^{\frac{1}{2}}, y''''(\frac{1}{2}) = e^{\frac{1}{2}}, y'''''(\frac{1}{2}) = e^{\frac{1}{2}}.$$

The exact solution is $y(x) = e^x$. Can be written eq.(3.5.1)

$$Ly = e^{-x}y^2(x), \tag{3.5.2}$$

from an operator (2.1), when m=0,n=4, we obtain

$$L(.) = x^{-1}\frac{d^2}{dx^2}x^3\frac{d}{dx}x^{-2}\frac{d^3}{dx^3}(.),$$

for three inverse operators under one of the following conditions, respectively

$$L^{-1}(.) = \int_0^x\int_0^x\int_0^x x^2\int_1^x x^{-3}\int_0^x\int_0^x x(.)\ dxdxdxdxdxdx,$$
$$L^{-1}(.) = \int_1^x\int_1^x\int_1^x x^2\int_1^x x^{-3}\int_1^x\int_0^x x(.)\ dxdxdxdxdxdx,$$
$$L^{-1}(.) = \int_{\frac{1}{2}}^x\int_{\frac{1}{2}}^x\int_{\frac{1}{2}}^x x^2\int_{\frac{1}{2}}^x x^{-3}\int_{\frac{1}{2}}^x\int_{\frac{1}{2}}^x x(.)\ dxdxdxdxdxdx.$$

Applying L^{-1}, to both sides of (3.5.2) and using the boundary condition respectively, gives

$$y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5 + L^{-1}e^{-x}y^2,$$
$$y(x) = 1.03 + 0.943x + 0.545x^2 + 0.143x^3 + 0.0425x^4 + 0.0135x^5 + L^{-1}e^{-x}y^2,$$
$$y(x) = 0.997 + 1.021x + 0.473x^2 + 0.172x^3 + 0.0345x^4 + 0.0344x^5 + L^{-1}e^{-x}y^2,$$

employing ADM for y^2, as yield

$$\sum_{n=0}^{\infty} y_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5 +$$
$$L^{-1}\sum_{n=0}^{\infty} e^{-x}A_n, n \geq 0,$$
$$\sum_{n=0}^{\infty} y_n(x) = 1.03 + 0.943x + 0.545x^2 + 0.143x^3 + 0.0425x^4 + 0.0135x^5 +$$
$$L^{-1}\sum_{n=0}^{\infty} e^{-x}A_n, n \geq 0,$$
$$\sum_{n=0}^{\infty} y_n(x) = 0.997 + 1.021x + 0.473x^2 + 0.172x^3 + 0.0345x^4 + 0.0344x^5 +$$
$$L^{-1}\sum_{n=0}^{\infty} e^{-x}A_n, n \geq 0,$$

the components for $y_n(x)$ introduces the recursive relation, respectively

$$y_0 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5,$$
$$y_0 = 1.03 + 0.943x + 0.545x^2 + 0.143x^3 + 0.0425x^4 + 0.0135x^5,$$
$$y_0 = 0.997 + 1.021x + 0.473x^2 + 0.172x^3 + 0.0345x^4 + 0.0344x^5,$$
$$y_{n+1} = L^{-1}e^{-x}A_n, n \geq 0,$$

applying Adomain polynomial A_n, for the non-linear term y^2, when for n=0,1, gives

$$A_0 = y_0^2,$$

$$A_1 = 2y_0 y_1,$$

the first components respectively, as follows

$$y_1 = -0.0036338\,x^5 + 0.00138889\,x^6 + 0.000198413\,x^7 + ... + 2.15102\,10^{-12}\,x^{20},$$
$$y_2 = 2.57621\,10^{-7}\,x^5 - 2.18483\,10^{-8}\,x^{11} + 4.17535\,10^{-9}\,x^{12} + ... + 8.35363\,10^{-13}\,x^{20},$$

$$y_1 = -0.00508562 + 0.0225422\,x - 0.036792\,x^2 + 0.0237785\,x^3 + ... + 8.28423\,10^{-14}\,x^{20},$$

$$y_2 = 2.31144\,10^{-6} - 0.0000113257\,x + 0.0000214098\,x^2 + ... + 1.99707\,10^{-13}\,x^{20},$$

$$y_1 = 0.0000334423 - 0.000396757\,x + 0.00195246\,x^2 + ... + 1.48474\,10^{-12}\,x^{20},$$
$$y_2 = 1.61882\,10^{-12} - 3.87397\,10^{-11}\,x + 4.24644\,10^{-10}\,x^2 + ... + 8.16771\,10^{-13}\,x^{20},$$

The solution in a series from is following, respectively

$$y(x) = y_0 + y_1 + y_2 = 1 + x + 0.5x^2 + 0.1667x^3 + 0.04167x^4 + ... + 4.38942\,10^{-36}\,x^{47},$$
$$y(x) = y_0 + y_1 + y_2 = 1.02492 + 0.965531\,x + 0.508229\,x^2 + ... + 2.9159\,10^{-37}\,x^{47},$$
$$y(x) = y_0 + y_1 + y_2 = 0.997033 + 1.0206\,x + 0.474952\,x^2 + ... + 4.82443\,10^{-36}\,x^{47},$$

Table 3.5.1.1. The comparison between Exact solution and MADM for under three conditions

x	Exact solution	MADN at the first condition	Absolute Error	MADM at the second condition	Absolute Error	MADM at the third condition	Absolute Error
0.0	1.00000	1.00000	0.00000	1.02492	0.02492	0.99703	0.002967
0.1	1.10517	1.10517	0.00000	1.12672	0.02155	1.10401	0.00116
0.2	1.22140	1.22139	0.00001	1.23976	0.01836	1.22156	0.00016
0.3	1.34986	1.34975	0.00011	1.36518	0.01532	1.35088	0.00102
0.4	1.49182	1.49136	0.00046	1.50429	0.01247	1.49332	0.0015
0.5	1.64872	1.64731	0.00141	1.65850	0.00978	1.65048	0.00176
0.6	1.82212	1.81860	0.00352	1.82937	0.00725	1.82418	0.00206
0.7	2.01375	1.00614	0.00761	2.01865	0.00490	2.01653	0.00278
0.8	2.22554	2.21070	0.01484	2.22825	0.00271	2.22999	0.00445
0.9	2.45960	2.43285	0.02675	2.46025	0.00065	2.46738	0.00778
1.0	2.71828	2.67298	0.04530	2.71700	0.00128	2.73194	0.01366

We will study the same example at m=1, give under one of the following condition

$$y(0) = 1, y'(0) = 1, y''(0) = 1, y'''(0) = 1, y'''''(0) = 1, y'''(1) = e,$$

$$y(1) = e, y'(1) = e, y''(1) = e, y'''(1) = e, y''''(1) = e, y'''''(0) = 1,$$

$$y(\frac{1}{2}) = e^{\frac{1}{2}}, y'(\frac{1}{2}) = e^{\frac{1}{2}}, y''(\frac{1}{2}) = e^{\frac{1}{2}}, y'''(\frac{1}{2}) = e^{\frac{1}{2}}, y''''(\frac{1}{2}) = e^{\frac{1}{2}}, y'''''(\frac{1}{2}) = e^{\frac{1}{2}}.$$

The exact solution is $y(x) = e^x$. Can be written eq.(3.5.1)

$$Ly = e^{-x}y^2(x), \tag{3.5.3}$$

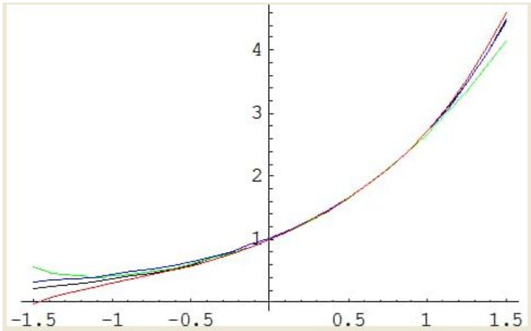

Fig. 3.5.1.1. Comparison between Exact solution and MADM under three conditions, respectively

from an operator (2.1), when m=1,n=4, we obtain

$$L(.) = \frac{d}{dx}x^{-1}\frac{d}{dx}x^{2}\frac{d}{dx}x^{-1}\frac{d^3}{dx^3}(.),$$

for three inverse operators under one of the following conditions, respectively

$$L^{-1}(.) = \int_0^x\int_0^x\int_0^x x\int_1^x x^{-2}\int_0^x x\int_0^x (.)\ dxdxdxdxdxdx.$$

$$L^{-1}(.) = \int_1^x\int_1^x\int_1^x\int_1^x x^{-2}\int_1^x x\int_0^x (.)\ dxdxdxdxdxdx.$$

$$L^{-1}(.) = \int_{\frac{1}{2}}^x\int_{\frac{1}{2}}^x\int_{\frac{1}{2}}^x x\int_{\frac{1}{2}}^x x^{-2}\int_{\frac{1}{2}}^x x\int_{\frac{1}{2}}^x (.)\ dxdxdxdxdxdx.$$

Applying L^{-1}, to both sides of (3.5.3) and using the boundary condition respectively, gives

$$y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + 0.05075x^4 + \frac{1}{120}x^5 + L^{-1}e^{-x}y^2,$$

$$y(x) = 1.0119 + 0.947x + 0.596x^2 + 0.083x^3 + 0.0717x^4 + 0.0083x^5 + L^{-1}e^{-x}y^2,$$

$$y(x) = 1.01127 + 0.916x + 0.755x^2 + 0.0172x^3 + 0.206x^4 + 0.01375x^5 + L^{-1}e^{-x}y^2,$$

employing ADM for y^2, as yield

$$\sum_{n=0}^{\infty} y_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + 0.05075x^4 + \frac{1}{120}x^5 +$$

$$L^{-1}\sum_{n=0}^{\infty} e^{-x}A_n, n \geq 0,$$

$$\sum_{n=0}^{\infty} y_n(x) = 1.0119 + 0.947x + 0.596x^2 + 0.083x^3 + 0.0717x^4 + 0.0083x^5 +$$

$$L^{-1}\sum_{n=0}^{\infty} e^{-x}A_n, n \geq 0,$$

$$\sum_{n=0}^{\infty} y_n(x) = 1.01127 + 0.916x + 0.755x^2 + 0.0172x^3 + 0.206x^4 + 0.01375x^5 +$$

$$L^{-1} \sum_{n=0}^{\infty} e^{-x} A_n, n \geq 0,$$

the components for $y_n(x)$ introduces the recursive relation, respectively

$$y_0 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + 0.05075x^4 + \frac{1}{120}x^5,$$

$$y_0 = 1.0119 + 0.947x + 0.596x^2 + 0.083x^3 + 0.0717x^4 + 0.0083x^5,$$

$$y_0 = 1.01127 + 0.916x + 0.755x^2 + 0.0172x^3 + 0.206x^4 + 0.01375x^5,$$

$$y_{n+1} = L^{-1} e^{-x} A_n, n \geq 0,$$

the first components respectively, as follows

$$y_1 = -0.00909843\, x^4 + 0.00138889\, x^6 + 0.000198413\, x^7 + ... + 3.27175\, 10^{-19}\, x^{25},$$

$$y_2 = 3.35831\, 10^{-6}\, x^4 - 1.2035\, 10^{-7}\, x^{10} + ... + 1.13674\, 10^{-18}\, x^{25},$$

$$y_1 = -0.0111946 + 0.0530443\, x - 0.0976931\, x^2 + 0.0845656\, x^3 + ... + 1.46188\, 10^{-17}\, x^{24},$$

$$y_2 = 0.0000324587 - 0.00016184\, x + 0.000321221\, x^2 + ... + 9.59612\, 10^{-17}\, x^{25},$$

$$y_1 = 0.0000346808 - 0.000410713\, x + 0.0020165\, x^2 + ... + 8.15701\, 10^{-18}\, x^{25},$$

$$y_2 = 1.70115\, 10^{-12} - 4.06568\, 10^{-11}\, x + 4.45143\, 10^{-10}\, x^2 + ... + 4.57668\, 10^{-16}\, x^{25},$$

The solution in a series from is following, respectively

$$y(x) = y_0 + y_1 + y_2 =$$

$$y(x) = 1 + x + 0.5x2 + 0.1667x^3 + 0.0416549\, x^4 + 0.00833x^5 + ... + 1.46391\, 10^{-18}\, x^{25},$$

$$y(x) = 1.00074 + 0.999882\, x + 0.540128\, x^2 + 0.167252\, x^3 + ... + 9.59612\, 10^{-17}\, x^{25},$$

$$y(x) = 1.0113 + 0.915589\, x + 0.757017\, x^2 + 0.166765\, x^3 + ... + 4.49511\, 10^{-16}\, x^{25},$$

Table 3.5.1.2. The comparison between exact solution and MADM for under conditions

x	Exact solution	MADN at the first condition	Absolute Error	MADM at the second condition	Absolute Error	MADM at the third condition	Absolute Error
0.0	1.00000	1.00000	0.00000	1.00074	0.00074	1.01130	0.01130
0.1	1.10517	1.10517	0.00000	1.1063	0.001130	1.11062	0.00545
0.2	1.22140	1.22140	0.00000	1.22372	0.00232	1.22638	0.00498
0.3	1.34986	1.34986	0.00000	1.35417	0.00431	1.36037	0.01051
0.4	1.49182	1.49182	0.00000	1.49888	0.00706	1.51489	0.02307
0.5	1.64872	1.64872	0.00000	1.65923	0.01051	1.69282	0.04410
0.6	1.82212	1.82212	0.00000	1.83669	0.01457	1.89759	0.75470
0.7	2.01375	2.01375	0.00000	2.03283	0.01908	2.13319	0.11944
0.8	2.22554	2.22554	0.00000	2.24936	0.02382	2.40422	0.17868
0.9	2.45960	2.45960	0.00000	2.48812	0.02852	2.71589	0.25629
1.0	2.71828	2.71827	0.00001	2.75110	0.03282	3.07406	0.35578

We note the table above for the condition one, we got the exact solution and the another conditions, we got the approximate solutions for the exact solution. Therefore the method is very good and effective.

From the Tables 3.5.1.1, 3.5.1.2 and the Figs. 3.5.1.1, 3.5.1.2, we noticed the convergence and we obtain exact solutions, and thus the method is very useful and active to solve from high-order.

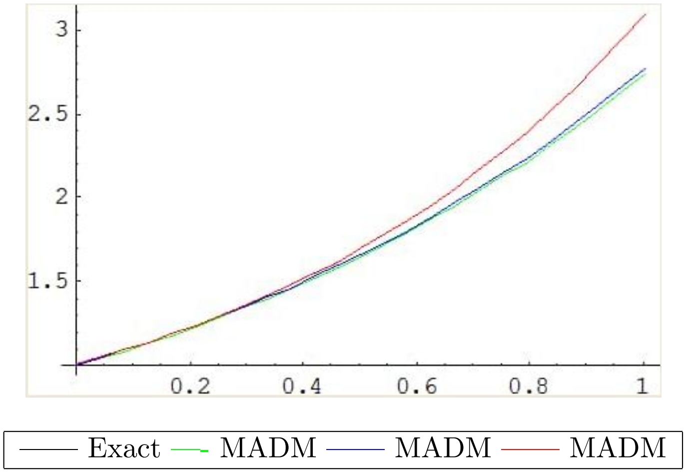

Fig. 3.5.1.2. Comparison between exact solution and MADM under three boundary conditions, respectively

4 Conclusion

This method is an active, useful and effective to get the approximate solutions. Through tables and figures are the previous illustrations of third-order boundary value problems into higher-order, we noticed that we reach the approximate solution and more than the exact solution. We found it highly efficient and it can be developed to be used to find solutions to develop the differential operator of the inverse operator by boundary conditions in general.

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Competing Interests

Authors have declared that no competing interests exist.

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