



## Finite Deformation of Internally Pressurized Spherical Compressible Rubber-Like Material

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### Authors' contributions

This work was carried out in collaboration between both authors. Author PEU designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author ENE managed the analyse of the study. Both authors read and approved the final manuscript.

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## Abstract

The finite deformation of a compressible internally pressurized spherical synthetic rubber-like material governed by Levinson and Burgess strain energy function is analysed. The analysis led to a second order nonlinear ordinary differential equation for the determination of stresses and displacements. Analytic solution is found impossible for now, hence, the solution is sought numerically using shooting method on mathematica and collocation method. The result of the two schemes were statistically compared using t-test to determine which method is better. Results obtained from the t-test is 1.0692 for the calculated value is less than the table value of 1.725 and since the p-value is greater than 0.05, this shows that the two methods has no significant difference. We conclude the two methods are similar.

*Keywords: Pressure; Levinson and Burgess; synthetic; mathematics; spherical; symmetric; collocation.*

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## 1 Introduction

The theory of materials subjected to large deformations often nonlinear and nonlinear materials have been found to be more realistic in practical terms, especially in engineering field and construction firms. The nature of mathematical models in theory of elasticity make the derived equations very tough if not impossible to obtain analytical or closed form solution to the problem. In the modeling of any hyper-elastic material, the major focus is on selecting the proper constitutive relation. Rubber is not just about the original natural rubber but also referred to any material that has same mechanical properties and they are in other words said to be rubber-like materials.

In our work, we concentrated on synthetic rubber which is made from petroleum and it is grouped as an artificial rubber. The synthetic rubber can be deformed without being damaged and the original shape is preserved after being stretched. This type of rubber is man-made and has much importance over natural rubber due to its superiority in performance. It is used often than natural rubber in most industrialized nations especially in producing car tires, medical equipment, machinery belts and moulded parts. Synthetic rubbers are said to be elastomers. Our choice of material is based on the fact that a lot has been achieved on natural rubbers but not much has been achieved in synthetic rubber materials due to the difficult nature of its model equations. For a rubber-like material, compressibility is an important physical property constantly needed in practical applications and in calculations that relates to mechanical properties. This is because rubber-like materials exhibit a highly nonlinear behaviour and can be applied to rubber and many other polymeric materials which are considered to be isotropic and hyper-elastic material. We are most concerned about the deformations of Levinson-Burgess proposed strain energy function which is said to be compressed. The literature review gives us a huge insight of work done in this area of elasticity. The strain energy function,  $W$ , as proposed by Levison & Burgess is given as:

$$W = \frac{\mu_0}{2} \left[ f(J_1 - 3) + (1 - f)(J_2 - 3) + 2(1 - 2f)(J_3 - 1) + 2f + \frac{4v - 1}{1 - 2v}(J_3 - 1)^2 \right] \quad (1.1)$$

where  $J_1 = I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ ,  $J_2 = \frac{I_2}{I_3} = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2}$  and  $J_3 = \sqrt{I_3} = \lambda_1 \lambda_2 \lambda_3$   
 $I_1$ ,  $I_2$  and  $I_3$  are the principal invariants of the Cauchy-Green stress tensor while  $\lambda_1, \lambda_2, \lambda_3$  are the eigenvalues.

$f$  and  $v$  are the material constants and poisson ratio respectively while  $\mu_0$  is the ground state shear modulus.

According to Levinson & Burgess, they showed in an experimental result that for highly compressible polyurethane foam rubber,  $f = 0$  and  $v = 0.25$  will reduce equation(1.1) to Blatz-Ko generalized strain energy function.

If  $f = 0$  and  $v = 0.5$ , equation(1) reduces to the Neo-Hookean strain energy function. Similarly, Mooney-Rivlin material for incompressible material is obtained as  $I_3 = 1$  when  $f = 1$  and  $v = 0.5$ . In compressible tests and constitutive models for the slight compressibility of elastic rubber-like materials, Horgan and Murphy [1] used the Mooney-Rivlin material for volumetric test which involves compression in the axial direction of lubricated cylindrical specimens within a rigid annulus. Their analysis is based on behavioral observation of slightly compressible rubbers undergoing volumetric testing.

### 1.1 Literature Review

The failure of spherical bodies undergoing internal pressure has been on the increase. Research works are on the increase of how best to determine the different behaviour of such materials. This research work is set to determine the behaviour of such bodies governed by the strain energy functions

proposed by Levinson-Burgess under internal pressure. This is mainly because this model governs bodies composed of synthetic rubber. This work is limited to isotropic hyper-elastic compressible synthetic material of a Levinson and Burgess strain energy function. However, the frame work can accommodate other strain energy functions and the resulting model equations may be solved by same method. According to Kao et al [2], Most of the starting point for modeling of various kinds of elastomers is the strain energy function. Khajehasaeid et al [3] developed a strain energy function for isotropic rubbers which satisfies all properties of hyperelastic model. The model contains a deformation mode-independent properties and the work by Hoss [4] on a new constitutive model for rubber-like materials reviews the different strain energy functions for several constitutive models focusing on incompressible elastomers. They proposed a new family of hyperelastic models and the strain energy functions retains both terms of the stiffening and that representing the characteristic oscillation in the stress versus strain curve undergoing small strains. This was also similar to the paper by Pence and Gou [5] who considered three different compressible versions of the conventional incompressible neo-Hookean material model. The three versions critically x-rayed the differences with respect to each other by use of neo-Hookean strain energy function. Their aim is to exhibit these differences. The research by Kanner and Horgan [6] on extension and torsion of strain-stiffening discussed about the effect on the response of solid circular cylinders in the combined deformation of torsion layered on axial extension. The result of axial force required to maintain pure torsion is compressive for the models considered.

Moreira et al [7] compared two types of deformation using experimental and theoretical methods. Result showed that simple shear cannot be considered as pure shear combined with a rotation when undergoing large deformation. It is a fact that rubber-like materials undergo large deformation and nonlinear upon loading as they return to the initial configuration after the removal of load. Hossain and Steinman [8] used phenomenological and micromechanical motivated network models for nearly incompressible hyperelastic polymeric materials in their paper. They used finite element framework for the solution of boundary value problem as derived in their work.

Ali et al [9] reviewed different classical continuum mechanics models for incompressible and isotropic materials dependent on strain energy potential which compares to neo-Hookean, Yeoh, Mooney-Rivlin and Ogden models in predicting uniaxial deformation. Horgan [10] also reviewed some of the numerous developments, extensions and widespread application resulting from not just this paper but papers in rubber elasticity and even biomechanics of soft biomaterials. The study by Jongmin and Dirk [11] specifically outlined the experiments carried out at different strain rates on continuous loading and unloading to characterize the deformation behaviour of polyurea under compressive loading. They developed a new model which predicts the response under monotonic loading given wide range of strain rates and the result agreed well with the experimental result. Blatz and Ko [12] in their work proposed a strain energy function which they called "Standard" strain energy function but Levinson and Burgess discovered that there were certain limitations. First, in the limit of incompressibility, they standard strain energy function cannot represent Mooney-Rivlin or Neo-Hookean material and secondly, It is not a capable strain energy function for an isotropic material. Based on these facts, Levinson and Burgess introduced the simplest rational polynomial strain energy function as shown in equation(1.1). Hence, the experimental value shows that for a synthetic rubber-like material,  $f = 1$  and  $v = 0.46$ , we obtain the strain energy function for a compressible material. At this point, equation(1.1) reduces to;

$$W = \frac{\mu_0}{2} \left[ I_1 - 27\sqrt{I_3} + \frac{25}{2}I_3 + \frac{23}{2} \right] \quad (1.2)$$

Sang et al [13] worked on rubber tubes under pressure undergoing large deformation and present a specific nonlinear elastic behaviour. They established a connection between internal pressure and the internal volume ratio.

Horgan & Murphy [14, 15, 16] did similar work to that of Levinson & Burgess on Compression tests and constitutive models for the slight compressibility of elastic rubber-like material. They investigated the role played by classical simple shear in nonlinear elasticity. They also determined the hydrostatic pressure for the particular case of neo-Hookean material and different stress distributions are compared and contrasted.

Fosdick et al [10] studied about toroidal twist-like bifurcations for an isotropic Levinson-Burgess compressible elastic tube subjected to pure circular shear. They applied a novel effective method based on magnus expansion to analyze the bifurcation problem thereby evaluating the critical load. Chung et al [17] in their study obtained the initiation of a localized shear bifurcation. They also obtained a maximum pressure which is relative to the bifurcation. Observation shows that when the ratio of the outer undeformed radius to the inner radius is larger than the critical value, the shear bifurcation occurs before the maximum pressure, while the reverse is true when this ratio is smaller than the critical value. This study specifically considered the strain energy function of Blatz-Ko material. Closed form analytic solutions were obtained for both the cylindrical and spherical deformations.

The work by Levinson and Burgess compared other strain energy functions with their proposed strain energy function by using the poisson ratio and material constant to determine whether the different behaviours are due to compressibility or a certain choice of compressible material. They compared three constitutive relations due to Blatz, Blatz-Ko and their work to predict different behaviour. We therefore, decided to extend their work considering an internally pressurized spherical rubber-like material deformation. We obtained the appropriate second order nonlinear boundary value problem where we solved for the displacements and stresses of the proposed material of Levinson and Burgess. The problem was solved using two different method just to validate the authenticity by using t-test to test for significant differences. This work can easily be applied to solve the problem of Car tyre where the stresses and displacements can be considered when the tyre undergoes inflation.

## 2 Materials and Methods

### 2.1 Spherical Polar Coordinates

Chung et al [17] Considering the spherical deformation of a hollow sphere where the deformation takes the point with the spherical polar coordinates  $(R, \Theta, \Phi)$  in the undeformed region to the point  $(r, \theta, \phi)$  in the deformed region.

$$r = r(R) \quad a \leq R \leq b \tag{2.1}$$

$$\theta = \Theta \quad 0 \leq \Theta \leq \pi \tag{2.2}$$

$$\phi = \Phi \quad 0 \leq \Phi \leq 2\pi \tag{2.3}$$

$$\frac{dr}{dR} > 0 \tag{2.4}$$

$$\tag{2.5}$$

$r(R)$  is to be determined and  $F = \text{diag}(\frac{dr}{dR}, \frac{r}{R}, \frac{r}{R})$

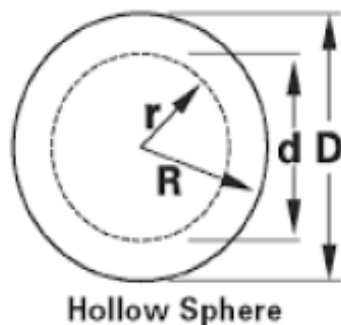


Fig. 1. Hollow Sphere

## 2.2 Development of Field Equation

### 2.2.1 Principal stretches (Eigenvalues)

$$\lambda_1 = \frac{dr}{dR}, \lambda_2 = \lambda_3 = \frac{r}{R}$$

### 2.2.2 Cauchy Green Right Tensor

$$\bar{C} = \bar{F}^T \bar{F} = \begin{bmatrix} \left(\frac{dr}{dR}\right)^2 & 0 & 0 \\ 0 & \left(\frac{r}{R}\right)^2 & 0 \\ 0 & 0 & \left(\frac{r}{R}\right)^2 \end{bmatrix}$$

### 2.2.3 Cauchy stresses

$$\begin{aligned} \sigma_{ii} &= \frac{\lambda_i}{\lambda_1 \lambda_2 \lambda_3} \frac{\partial W}{\partial \lambda_i}, \text{ for } i = 1, 2, 3 \\ \sigma_{11} = \sigma_{rr} &= \frac{\mu_0}{2} \left[ \frac{2r'}{r^2} R^2 + \frac{25r'}{R^2} r^2 - 27 \right] \\ \sigma_{22} = \sigma_{33} = \sigma_{\theta\theta} = \sigma_{\phi\phi} &= \frac{\mu_0}{2} \left[ \frac{2}{r'} + \frac{25r'}{R^2} r^2 - 27 \right] \\ \sigma_{r\theta} = \sigma_{\theta r} = \sigma_{\theta\phi} = \sigma_{\phi\theta} = \sigma_{r\phi} = \sigma_{\phi r} &= 0 \end{aligned}$$

## 2.3 Equilibrium Equations in Spherical Coordinates System

Mohammed & Rahman(2014) in their work on a simplified method for deriving equilibrium equations in solid continuous systems applied what they termed direct method to obtain the equation of equilibrium:

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi r}}{\partial \phi} + \frac{1}{r} (2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{\theta r} \cot \theta) + f_r &= 0 \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{r} (3\sigma_{r\theta} + (\sigma_{\theta\theta} - \sigma_{\phi\phi}) \cot \theta) + f_\theta &= 0 \\ \frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r} (3\sigma_{r\phi} + (\sigma_{\theta\theta} - \sigma_{\theta\phi} \cot \theta)) + f_\phi &= 0 \end{aligned}$$

Recall the Cauchy first law of continuum mechanics where there is no body force;

$$\text{div} \sigma = 0$$

where  $f_r$ ,  $f_\theta$  and  $f_\phi$  represent the body forces which are all zero. We are to consider the spherical coordinates taking into cognizance the fact that we are dealing with symmetric deformations. We

transform the equilibrium equation, by applying the cauchy stresses  $\sigma_{rr}$ ,  $\sigma_{\theta\theta}$ ,  $\sigma_{\phi\phi}$  and  $\sigma_{r\theta} = \sigma_{\theta r} = \sigma_{\theta\phi} = \sigma_{\phi\theta} = \sigma_{r\phi} = \sigma_{\phi r} = 0$  into the equilibrium equation;

$$\frac{d\sigma_{rr}}{dR} + \frac{2r'}{r} [\sigma_{rr} - \sigma_{\theta\theta}] = 0,$$

where  $r' = \frac{dr}{dR}$

We then solve the derivative to obtain the model equation for the spherical deformation of a synthetic rubber as;

$$\begin{aligned} \sigma_{rr} &= \left[ \frac{2r'}{r^2} R^2 + \frac{25r'}{R^2} r^2 - 27 \right] \\ \frac{d(\mu(\frac{r'R^2}{r^2} + \frac{25r'r^2}{R^2} - 27))}{dR} + \mu \frac{2r'}{r} \left[ \frac{r'R^2}{r^2} + \frac{25r'r^2}{R^2} - 27 - (\frac{2}{r'} + \frac{25r'r^2}{R^2} - 27) \right] &= 0, \\ \mu \left[ \frac{d(\frac{2r'R^2}{r^2} + \frac{2r'r^2}{R^2} - 27)}{dR} + (\frac{4r'^2 R}{r^3} - \frac{4}{r}) \right] &= 0, \\ \frac{\mu}{2} \left[ \frac{2r''R - 2rr' - 4(r')^2 R}{r^3} + \frac{25r''r^2 R + 50r'^2 r R - 50r'r^2}{R^3} + (\frac{4r'^2 R}{r^3} - \frac{4}{r}) \right] &= 0 \\ \frac{2rr''R^4 + 2rr'R^3 - 4r'^2 R^4 + 25r''r^5 R + 50r'^2 r^4 R - 50r'r^5 + 4r'^2 R^4 - 4r^2 R^3}{r^3 R^3} &= 0 \end{aligned}$$

Note that  $\mu \neq 0$  Therefore, the above equation becomes;

$$\begin{aligned} r''(2rR^4 + 25r^5 R) + 50r'^2 r^4 R + r'(2rR^3 - 50r^5) - 4r^2 R^3 &= 0, \\ (2R^4 + 25r^4 R) \frac{d^2 r}{dR^2} + 50r^3 R (\frac{dr}{dR})^2 + (4R^3 - 50r^4) \frac{dr}{dR} - 2rR^3 &= 0 \end{aligned}$$

## 2.4 Boundary Value Model Equation of Spherically Symmetric Deformation

$$(2R^2 + 25r^4 R) \frac{d^2 r}{dR^2} + 50r^3 R (\frac{dr}{dR})^2 + (4R^4 - 50r^4) \frac{dr}{dR} - 2rR^3 = 0$$

Applying the conditions of a hollow sphere on the radial stress, we obtain the boundary conditions as;

$$\begin{aligned} \sigma_{11} = \sigma_{rr} &= \frac{\mu_0}{2} \left[ \frac{2r'}{r^2} R^2 + \frac{25r'}{R^2} r^2 - 27 \right] \\ (\sigma_{rr})_{R=a} = -\rho &\implies r'(a) = \frac{a^2 r^2(a)(27\mu_0 - 2\rho)}{\mu_0(2a^4 + 25r^4(a))} \\ (\sigma_{rr})_{R=b} = 0 &\implies r'(b) = \frac{27b^2 r^2(b)}{2b^4 + 25r^4(b)} \end{aligned}$$

## 3 Solutions for Spherical Symmetric Deformation

We now proceed to investigate the displacements and stresses of the compressible rubber-like material undergoing internal pressure. First, we consider the derived equation of spherically symmetric deformations which we solved numerically by the help of Mathematica(ode45 solver) and collocation method. The boundary value problem is a second-order nonlinear ordinary differential equations which at this moment has no analytical solution even after applying several techniques. The load applied is the same at every height and the load considered here is the pressure which is applied at a constant rate at the inner surface. This applied load generates stresses and displacements between inner and outer surface.

Now we solve the boundary value problem given as;

$$(2R^2 + 25r^4R) \frac{d^2r}{dR^2} + 50r^3R \left(\frac{dr}{dR}\right)^2 + (4R^4 - 50r^4) \frac{dr}{dR} - 2rR^3 = 0$$

Applying the conditions of a hollow sphere on the radial stress, we obtain the boundary conditions as;

$$\begin{aligned} \sigma_{11} &= \sigma_{rr} = \frac{\mu_0}{2} \left[ \frac{2r'}{r^2} R^2 + \frac{25r'}{R^2} r^2 - 27 \right] \\ (\sigma_{rr})_{R=a} &= -\rho \implies r'(a) = \frac{a^2 r^2(a)(27\mu_0 - 2\rho)}{\mu_0(2a^4 + 25r^4(a))} \\ (\sigma_{rr})_{R=b} &= 0 \implies r'(b) = \frac{27b^2 r^2(b)}{2b^4 + 25r^4(b)} \end{aligned}$$

### 3.1 Shooting Method

step 1: set  $h = \frac{b-a}{N}$   
 $K = 1$   
 $TK = \frac{\beta-\alpha}{b-a}$   
step 2: while ( $K \leq M$ ), do steps 3-10  
step 3: set  $w_{1,0} = \alpha$   
 $w_{2,0} = TK$   
 $U_1 = 0$   
 $U_2 = 1$  Step 4: For  $i = 1, \dots, N$  do steps 5 and 6 (The Runge-Kutta method for systems is applied in step 5 and step 6)  
step 5: set  $x = a + (i-1)h$   
step 6: set  $k_{1,1} = hw_{2,i-1}$ ;  
 $k_{1,2} = hf(x, w_{1,i-1} + w_{2,i-1})$ ;  
 $k_{2,2} = hf(x + \frac{h}{2}, w_{1,i-1} + \frac{1}{2}k_{1,1}, w_{2,i-1} + \frac{1}{2}k_{1,2})$ ;  
 $k_{3,1} = h(w_{2,i-1} + k_{2,2})$ ;  
 $k_{3,2} = hf(x + \frac{h}{2}, w_{1,i-1} + \frac{1}{2}k_{2,1}, w_{2,i-1} + \frac{1}{2}k_{2,2})$ ;  
 $k_{4,1} = h(w_{2,i-1} + k_{3,2})$ ;  
 $k_{4,2} = hf(x + h, w_{1,i-1} + k_{3,1}, w_{2,i-1} + k_{3,2})$ ;  
 $w_{1,i} = \frac{(w_{1,i-1} + (k_{1,1} + 2k_{2,1} + 2k_{3,1} + 2k_{4,1}))}{6}$ ;  
 $k'_{1,1} = hu_2$ ;  
 $k'_{1,2} = h[f_y(x, w_{1,i-1}, w_{2,i-2})u_1 + f'_{1y}(x, w_{1,i-1}, w_{2,i-1})u_2]$ ;  
 $k'_{2,1} = h[u_2 + \frac{1}{2}k'_{1,2}]$ ;  
 $k'_{2,2} = h[f_y(x + \frac{h}{2}, w_{1,i-1}, w_{2,i-2})(u_1 + \frac{1}{2}k'_{1,1}) + f'_{1y}(x + \frac{h}{2}, w_{1,i-1}, w_{2,i-1})(u_2 + \frac{1}{2}k'_{1,2})]$ ;  
 $k'_{3,1} = h[u_2 + \frac{1}{2}k'_{2,2}]$ ;  
 $k'_{3,2} = h[f_y(x + \frac{h}{2}, w_{1,i-1}, w_{2,i-2})(u_1 + \frac{1}{2}k'_{2,1}) + f'_{1y}(x + \frac{h}{2}, w_{1,i-1}, w_{2,i-1})(u_2 + \frac{1}{2}k'_{2,2})]$ ;  
 $k'_{4,1} = h[u_2 + k'_{3,2}]$ ;  
 $k'_{4,2} = h[f_y(x + h, w_{1,i-1}, w_{2,i-2})(u_1 + k'_{3,1}) + f'_{1y}(x + h, w_{1,i-1}, w_{2,i-1})(u_2 + k'_{3,2})]$ ;  
 $u_1 = u_1 + \frac{1}{6}[k'_{1,1} + 2k'_{2,1} + k'_{3,1} + k'_{4,1}]$ ;  
 $u_2 = u_2 + \frac{1}{6}[k'_{1,2} + 2k'_{2,2} + 2k'_{3,2} + k'_{4,2}]$   
step 8: for  $i = 0, 1, \dots, N$ ,  
set  $x = a + ih$   
output  $(x, w_{1,i}, w_{2,i})$   
step 9: Procedure is complete  
stop  
step 10 : set  $Tk = Tk - \left(\frac{w_{1,N} - \beta}{u_1}\right)$  Newton's method is applied to compute Tk. Where Tk is the slope of the straight line through  $(a, \alpha)$  and  $(b, \beta)$ .

This is the steps taken internally in mathematica to solve the problem using shooting method.

### 3.2 The Collocation Method

Exponentially fitted backward differentiation scheme for general second order differential equations derived via collocation method, with frequency  $w = 1, h = \frac{(b-a)}{N}$ ; where (a,b) is the interval of integration, N is number of subintervals and  $a = x_0 \mid x_1 \mid \dots \mid x_N = b$ .

$$y_{2+i} = \frac{1}{((-1+e^{hw})^2 w^2)} (f_i - 2e^{hw} f_i + e^{2hw} f_i - e^{hw} h^2 w^2 f_i - 2f_{1+i} + 4e^{hw} f_{1+i} - 2e^{2hw} f_{1+i} + h^2 w^2 f_{1+i} + e^{2hw} h^2 w^2 f_{1+i} + f_{2+i} - 2e^{hw} f_{2+i} + e^{2hw} f_{2+i} - e^{hw} h^2 w^2 f_{2+i} - w^2 y_i + 2e^{hw} w^2 y_i - e^{2hw} w^2 y_i + 2w^2 y_{1+i} - 4e^{hw} w^2 y_{1+i} + 2e^{2hw} w^2 y_{1+i}),$$

$$yp_i = \frac{-1}{6(-1+e^{hw})^2 h w^2} (-6f_i] + 6e^{hw} f_i - 6f_i - 5e^{hw} h^2 w^2 f_i + 2^{2hw} h^2 w^2 f_i + 12f_{1+i} - 12e^{hw} f_{1+i} + 12hw f_{1+i} + 5h^2 w^2 f_{1+i} + e^{2hw} h^2 w^2 f_{1+i} - 6f_{2+i} + 6e^{hw} f_{2+i} - 6hw f_{2+i} - 2h^2 w^2 f_{2+i} - e^{hw} h^2 w^2 f_{2+i} + 6w^2 y_i - 12e^{hw} w^2 y_i + 6e^{2hw} w^2 y_i - 6w^2 y_{1+i} + 12e^{hw} w^2 y_{1+i} - 6e^{2hw} w^2 y_{1+i}),$$

$$yp_{1+i} = \frac{1}{6(-1+e^{hw})^2 h w^2} (6f_i - 6e^{hw} f_i + 6e^{hw} h w f_i - 4e^{hw} h^2 w^2 f_i + e^{2hw} h^2 w^2 f_i - 12f_{1+i} + 12e h w f_{1+i} - 12e^{hw} h w f_{1+i} + 4h^2 w^2 f_{1+i} + 2e^{2hw} h^2 w^2 f_{1+i} + 6f_{2+i} - 6e^{hw} f_{2+i} + 6e^{hw} h w f_{2+i} - h^2 w^2 f_{2+i} - 2e^{hw} h^2 w^2 f_{2+i} - 6w^2 y_i + 12e^{hw} w^2 y_i - 6e^2 h w w^2 y_i + 6w^2 y_{1+i} - 12e^{hw} w^2 y_{1+i} + 6e^{2hw} w^2 y_{1+i}),$$

$$yp_{2+i} = \frac{-1}{6(-1+e^{hw})^2 h w^2} (-6f_i + 6e^{hw} f_i - 6e^{2hw} h w f_i + 7e^{hw} h^2 w^2 f_i + 2e^{2hw} h^2 w^2 f_i + 12f_{1+i} - 12e^{hw} f_{1+i} + 12e^{2hw} h w f_{1+i} - 7h^2 w^2 f_{1+i} - 11e^{2hw} h^2 w^2 f_{1+i} - 6f_{2+i} + 6E^{hw} f_{2+i} - 6e^{2hw} h w f_{2+i} - 2h^2 w^2 f_{2+i} + 11e^{hw} h^2 w^2 f_{2+i} + 6w^2 y_i - 12e^{hw} w^2 y_i + 6e^{2hw} w^2 y_i - 6w^2 y_{1+i} + 12e^{hw} w^2 y_{1+i} - 6e^{2hw} w^2 y_{1+i}$$

Graphical representation of the solution at N = 20

```
Plot[Evaluate[r[R] /. sols], {R, 0.2, 1}, AxesLabel -> {"R", "r(R)"}]
```

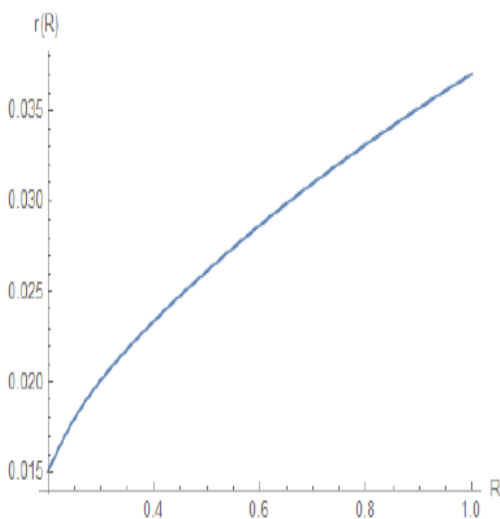


Fig. 2. Spherically symmetric deformation for N = 20



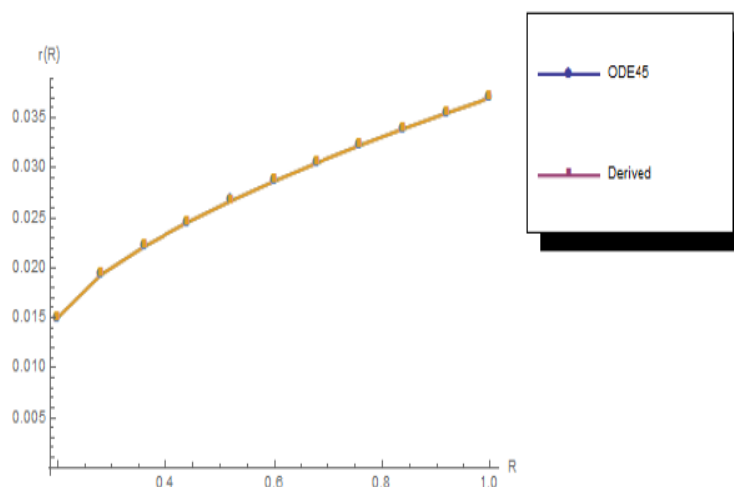


Fig. 3. Spherically symmetric deformation Vs. Collocation for N = 20

Table 1. Table for spherical shooting method Vs. Collocation method for N= 20

R	r(R)(shooting method)	r(R)( Collocation method)
0.2	0.015013	0.015012
0.24	0.017463	0.017461
0.28	0.019335	0.019333
0.32	0.020844	0.020842
0.36	0.022167	0.022165
0.4	0.023392	0.023390
0.44	0.024548	0.024546
0.48	0.025649	0.025647
0.52	0.026703	0.026700
0.56	0.027715	0.027713
0.6	0.028691	0.028688
0.64	0.029635	0.029632
0.68	0.030548	0.030545
0.72	0.031435	0.031432
0.76	0.032298	0.032294
0.8	0.033138	0.033134
0.84	0.033957	0.033953
0.88	0.034756	0.0347523
0.92	0.035537	0.035534
0.96	0.036302	0.036299
1.	0.037051	0.0370476

## 4 Conclusions

We considered a material that has Levinson-Burgess strain energy function. This is used because most strain energy functions are offshoot of this particular strain energy function. Several methods

of analytical approach to solve the resulting boundary value problem was sort but no solution was obtained at the moment. fortunately, a lot of softwares has been developed to handle such highly nonlinear second order ordinary differential equations with specific values of parameters. We were able to determine the maximum stress to be  $\sigma(r(1)) = -0.00035$  and the displacements of the resultant second order nonlinear ordinary differential equation with its mixed boundary condition for the spherical symmetric deformations. The deformation of radially inflated tyre was solved numerically at maximum pressures by the help of Mathematica (ode45 solver) algorithm which used shooting method internally and we compared result the with our derived results where we applied collocation method to solve for spherical problem.

Results show that the spherical symmetric deformation attains its maximum pressure at  $\rho = 0.5$  and when the material is inflated beyond that, the rubber rounds out and the top of the material will quickly wear out. There will be traction reduction which is responsible for the material to burst . From the table and graphical simulations, the result for shooting method as implemented internally by Mathematica agrees with the result from the derived collocation method for the same boundary value problems. This problem can be useful in determining the stresses and displacements of tyres undergoing internal pressure by way of inflation. In order to determine the authenticity of the methods, we employed t-test and we discovered that p-value has no significant difference, this means that the method are the same.

To further expand on this work, we can apply similar solution to the cylindrical deformation to obtain result using the two methods mentioned above.

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## Competing Interests

Authors have declared that no competing interests exist.

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## APPENDIX

### The SPERICAL PROBLEM

`a = 0.2; b = 1; μ = 0.46; ρ = 0.5; (*Initial conditions and parameters*)`

`coef1 = 2 r[R]^2 + 25 r[R]^4 R; coef2 = 50 r[R]^3 r'[R]^2 R; coef3 = (4 R^4 - 50 r[R]^4); coef4 = 2 r[R] R^3;`  
 (\*Coefficients of the dif. eqn.\*)

`bc1 =  $\frac{r[a]^2 a^2}{\mu} \left( \frac{27\mu - 2\rho}{2a^4 + 25r[a]^4} \right)$ ; bc2 =  $\frac{27r[b]^2 b^2}{2b^4 + 25r[b]^4}$ ;` (\*Left and Right Boundary conditions\*)

`Eqn = {coef1 r''[R] + coef2 + coef3 r'[R] - coef4 == 0}; (*The main differential Equation*)`

`bc = {r'[a] == bc1, r'[b] == bc2}; (*Boundary conditions*)`

### The "Shooting" method algorithm

`sols = Map[First[NDSolve[{Eqn, bc}, r, R, Method -> {"Shooting", "StartingInitialConditions" -> {r[0.2] == 1, r'[0.2] == 1}}]] &, {1}];`

### Table of Solutions at different mesh points

`Table[{R, Evaluate[r[R] /. sols]}, {R, 0.2, 1, 0.04}] // TableForm`

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