African Journal of Mathematics and Computer Science Research Vol. 5 (7), pp. 120-129, 9 April, 2012
Available online at http://www.academicjournals.org/AJMCSR
DOI: 10.5897/AJMCSR11.126
ISSN 2006-9731 © 2012 Academic Journals

Full Length Research Paper

The maximal allocated cost and minimal allocated benefit for interval data

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Accepted 29 August, 2011

This Paper deals with calculating the minimal and maximal shares of individuals or organizations based on different criteria. Suppose that players are selfish and the score for each criterion for a player is an interval. Each player makes any possible efforts to bring about his or her ideal condition. In this paper a new scheme to calculate the minimal allocated cost and the maximal allocated benefit for interval data is offered. In this scheme also the new models have been proposed for avoiding zero weight occurrence. Here ultimate allocated is achieved for each player with suitable coalition within several defined coalition.

Key words: Data envelopment analysis (DEA), game theory, and assurance region method.

INTRODUCTION

Consider *n* players each have *m* criteria for evaluating their competency or ability, which is represented by interval. For example, consider a usual classroom examination, the higher score for a criterion is, the better player is judged to perform that criterion. For example let the players be three students A, B and C, with three criteria, linear algebra, real analysis and numerical analysis are supposed to be variable in an interval. Now the problem is allocated a certain amount of fellowship in accordance to their score at these three criteria. All players are supposed to be selfish in the sense that they insist on their own advantage on the scores. However, they must reach a consensus in order to get the fellowship. This paper with using allocating and imputing the given benefit (Jahanshahloo et al., 2006) propose a new scheme for compute maximal allocated benefit and minimal allocated cost for players under the framework of game theory and data envelopment analysis (DEA) (Cooper et al. 2000). By this scheme the zero weight occurrence can be avoided. It also applies to determining the coalition with the ultimate benefit and least cost. The sections of this paper are as follows. In next sections, first the basic models are described and some properties of problem are proved, then extension of the basic model and introduce coalitions are discussed. Finally after a numerical example, conclusions and some remarks are present.

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BASIC MODELS OF THE GAME

We introduce the basic models and structures of the game based on Nakabayashi and Tone (2006).

Selfish behavior

Let $[x_{ij}^{1}, x_{ij}^{u}]$ be the score of player *j* in the criterion *i*, for $i = 1,...,m$ and $j = 1,...,n$ and $x_{ij}^{1} > 0, x_{ij}^{u} > 0$. It is assumed that the higher the interval score for a criterion is, the better player is judged to perform as regard to that criterion. Each person *k* has a right to choose two sets of nonnegative weights $w^k = (w_1^k,...,w_m^k)$ to the criteria that are most preferable to the player. Using the weight w^k, the relative scores of player *k* to the total score are defined as follows:

$$\frac{\sum_{i=1}^{m} w_i^{\ k} x_{ik}^{1}}{\sum_{i=1}^{m} w_i^{\ k} (\sum_{j=1}^{n} (x_{ij}^{1} + x_{ij}^{u}))}, \frac{\sum_{i=1}^{m} w_i^{\ k} x_{ik}^{u}}{\sum_{i=1}^{m} w_i^{\ k} (\sum_{j=1}^{n} (x_{ij}^{1} + x_{ij}^{u}))} \quad (1)$$

The denominator represents the total score of all players as measured by player *k* weight selection. While the numerators indicate player *k* self- evaluation as a lower and upper bound lower bound. Therefore, the statement (1) represents player *k* lower and upper relative importance in accordance to the x_{ik}^{1}, x_{ik}^{u} . We assume that the weighted scores are transferable. Player *k* wishes to maximize this ratio by selecting the most preferable weights w_k , for each of x_{ik}^{1}, x_{ik}^{u} as follows:

$$\begin{aligned} &\underset{w^k}{Max} \frac{\sum_{i=1}^{m} w_i^{\ k} x_{ik}^{1}}{\sum_{i=1}^{m} w_i^{\ k} (x_{ij}^{1} + x_{ij}^{u}))} \\ &s.t \quad w_i^{k} \geq 0 \quad (\forall i) \end{aligned} \quad (2)$$

$$\begin{aligned} &\underset{w^k}{Max} \frac{\sum_{i=1}^{m} w_i^{\ k} x_{ik}^{u}}{\sum_{i=1}^{m} w_i^{\ k} (x_{ij}^{1} + x_{ij}^{u}))} \\ &s.t \quad w_i^{k} \geq 0 \quad (\forall i) \end{aligned} \quad (3)$$

The motivation behind this problem is that the player *k* wishes to maximize his lower relative efficiency by (2), and upper relative efficiency by (3).As we can see in (2) the lower weighted sum of its record to the total weighted sum of all player records is maximized and in (3) the upper weighted sum of its record is maximized. We reformulate the problem, without losing generality. We normalize the data set X so that it is row – wise normalized, that is., $\sum_{j=1}^{n} x_{ij} = 1 \ (\forall i)$ (Charnes et al., 1978). We divide the row $(x_{i1},..,x_{in})$ by the row-sum

$$\sum_{j=1}^{n} (x_{ij}^{1} + x_{ij}^{u}) = 1, \ i = 1,...,m \quad (4)$$

Thus, using the Charnes-Cooper transformation scheme, the fractional programs (2) and (3) can be expressed using a linear programs follows:

$$\begin{aligned} c^1(k) &= Max \ \sum_{i=1}^{m} w_i^{\ k} x_{ik}^{1} \\ s.t \ & \sum_{i=1}^{m} w_i^{\ k} = 1 \\ & w_i^{\ k} \geq 0 \quad \forall i \end{aligned} \quad (5)$$

$$\begin{aligned} c^u(k) &= Max \ \sum_{i=1}^{m} w_i^{\ k} x_{ik}^{u} \\ s.t \ & \sum_{i=1}^{m} w_i^{\ k} = 1 \\ & w_i^{\ k} \geq 0 \quad \forall i \end{aligned} \quad (6)$$

After solving problems (5) and (6), if the optimal value of the problem (5) and (6) are $c^1(k)$ and $c^u(k)$ respectively, then $c^1(k) + c^u(k)$ may be considered as optimal value of the problem. Now the problem is to maximize the objectives (5) and (6) on the simplex $\sum_{i=1}^{m} w_i^{\ k} = 1$. Apparently, the optimal solution is given by assigning 1 to $w_{i(k)}^{k}$ and $w_{i(k')}^{\ k}$ for the criterion *i*(*k*) and $i(k')$ such that $x_{i(k)}^{1} = Max \{x_{ik}^{1} \mid i = 1,\ldots\ldots,m\}$ and $x_{i(k')}^{u} = Max\{x_{ik}^{u} \mid i = 1,.....,m\}$ respectively. Therefore, the optimal values will be as follows: ning 0 to the weight of remaining criteria. We denote this optimal value by *c*(*k*).

$$c^1(k) = x_{ik}^{1} \quad \text{and} \quad c^u(k) = x_{ik}^{u} \quad K=1,\ldots,n \quad (7)$$

$$\sum_{k=1}^{n} c^1(k) + c^u(k) \geq 1 \quad (8)$$

Theorem 1

Proof

Let the optimal weight for player *k* be $w_k^{1*} = (w_{1k}^{1*}...,w_{mk}^{1*})$ and $w_k^{u*} = (w_{1k}^{u*}...,w_{mk}^{u*})$, $w_{i(k)k}^{\ 1*} = 1$ and $w_{ik}^{\ 1*} = 0 (\forall i \neq i(k))$ and $w_{i(k')k}^{\ u*} = 1$ and $w_{ik}^{\ u*} = 0 (\forall i \neq i(k'))$ .Then we have

$$\begin{aligned} \sum_{k=1}^{n} c^1(k) + c^u(k) &= \sum_{k=1}^{n} c^1(k) + \sum_{k=1}^{n} c^u(k) = \sum_{k=1}^{n}\sum_{i=1}^{m} w_{i\ k}^{1*} \ x_{ik}^{1} \\ &+ \sum_{k=1}^{n}\sum_{i=1}^{m} w_{i\ k}^{u*} \ x_{ik}^{u} = \sum_{i=1}^{n} x_{i(k)k}^{1} + \sum_{i=1}^{n} x_{i(k')k}^{u} \geq \sum_{k=1}^{n} x_{1k}^{1} + \sum_{k=1}^{n} x_{1k}^{u} = 1 \end{aligned}$$

The inequality above follows from $x_{i(k)k}^{1} \geq x_{1k}^{1}$ and $x_{i(k')k}^{u} \geq x_{1k}^{u}$ and the last equality follows from the row – wise normalization. This theorem assert that, if each player sticks to his egoistic sense of value and insists on getting the portion of the benefit as designated by $c^1(k)$ and $c^u(k)$, the sum of shares usually exceeds 1 and hence $c^1(k) + c^u(k)$ cannot fulfill the role of division of the benefit. If eventually the sum of $c^1(k)$ and $c^u(k)$ turns out to be 1, all players will agree to accept the division $c^1(k)+c^u(k)$, since this is obtained by the players most preferable weight selection. The latter case will occur when all players have the same and common optimal weight selection, we have the following theorem.

Theorem 2

The equality $\sum_{k=1}^{n} c^1(k)+c^u(k)=1$ holds if and only if our data satisfies the condition $x^1_{1k}=x^1_{2k}=....=x^1_{mk}$ and $x^u_{1k}=x^u_{2k}=....=x^u_{mk}$, $\forall k=1,...,n$. That is, each player has the same score with respect to the *m* criteria.

Proof

The (if) part can be seen as follows. Since $c^1(k)=x^1_{1k}$ and $c^u(k)=x^u_{1k}$ for all *k* , we have:

$$\sum_{k=1}^{n} c^1(k)+c^u(k)=\sum_{k=1}^{n} c^1(k)+\sum_{k=1}^{n} c^u(k)=\sum_{k=1}^{n} x^1_{1k}+\sum_{k=1}^{n} x^u_{1k}=1$$

The (only if) part can be proved as follows. Suppose $x^1_{11}>x^1_{21}$ and $x^u_{11}>x^u_{21}$ then there must be column $h\neq 1$ and $h'\neq 1$ such that $x^1_{1h}<x^1_{2h}$ and $x^u_{1h}<x^u_{2h}$, otherwise the second row sum cannot attain 1. Thus we have $c^1(1)\geq x^1_{11}$, $c^u(1)\geq x^u_{11}$ and $c^1(h)\geq x^1_{2h}>x^1_{1h}$, $c^u(h')\geq x^u_{2h'}>x^u_{1h'}$, $c^1(j)\geq x^1_{1j}(\forall j\neq 1,h)$ $c^u(j)\geq x^u_{1j}(\forall j\neq 1,h')$. Hence it holds that

$$\sum_{k=1}^{n} c^1(k)+c^u(k)=\sum_{k=1}^{n} c^1(k)+\sum_{k=1}^{n} c^1(k)\geq \sum_{j=1,\neq h}^{n} x^1_{1j}+x^1_{2h}$$

$$+\sum_{j=1,\neq h'}^{n} x^u_{2j}+x^u_{2h'}>\sum_{j=1}^{n} x^1_{1j}+\sum_{j=1}^{n} x^u_{1j}=1$$

This leads to a contradiction. Therefore player1 must have the same score in all criteria. The same relation must hold for the other players. In the above case, only one criterion is needed for describing the game and the division proportional to this score is a fair division.

However, such situation might occur only in rare instances. In the majority of cases, we have $\sum_{k=1}^{n} c^1(k)+c^u(k)>1$.

Coalition with additive property

Let the coalition S be a subset of player set $N=(1,...,n)$. The record for coalition S is defined by

$$x^1_i(S)=\sum_{j\in S} x^1_{ij},\ x^u_i(S)=\sum_{j\in S} x^u_{ij}\ (i=1,...,m) \qquad (9)$$

These coalitions aim to maximize the out comes $c^1(S), c^u(S)$

$$c^1(S)=Max\sum_{i=1}^{m} w_i x^1_i(S)$$
$$s.t\ \sum_{i=1}^{m} w_i=1$$
$$w_i\geq 0\ \forall i \qquad (10)$$

$$c^u(S)=Max\sum_{i=1}^{m} w_i x^u_i(S)$$
$$s.t\ \sum_{i=1}^{m} w_i=1$$
$$w_i\geq 0\ \forall i \qquad (11)$$

The $c^1(S)$ and $c^u(S)$ with $c^1(\varphi)=0$ and $c^u(\varphi)=0$ defines a characteristic function of the coalition S . Thus this game is represented by (*N,c*).

Definition 1

A function f is called sub – additive if for any $S\subset N$ and $T\subset N$ with $S\bigcap T=\phi$ the following statement holds:

$f(S\bigcup T)\leq f(S)+f(T)$.

Definition 2

A function f called super – additive if for any $S\subset N$ and $T\subset N$ With $S\bigcap T=\phi$ the following statement holds: $f(S\bigcup T)\geq f(S)+f(T)$

Theorem 3

The characteristic function c is sub – additive, for any $S\subset N$ and $T\subset N$ with $S\bigcap T=\phi$ we have $c^1(S\bigcap T)+c^u(S\bigcap T)\leq c^1(S)+c^u(S)+c^1(T)+c^u(T)$

Proof

By renumbering the indexes, we can assume that $S=\{1,...,h\}, T=\{h+1,...,k\}$ and $S\bigcup T=\{1,...,k\}$. For these sets, it holds that

$$c^1(S \cup T) + c^u(S \cup T) = \underset{i}{Max} \sum_{j=1}^{k} x^1_{ij} + \underset{i}{Max} \sum_{j=1}^{k} x^u_{ij} \leq \underset{i}{Max} \sum_{j=1}^{h} x^1_{ij} +$$
$$\underset{i}{Max} \sum_{j=h+1}^{k} x^1_{ij} + \underset{i}{Max} \sum_{j=1}^{h} x^u_{ij} + \underset{i}{Max} \sum_{j=h+1}^{k} x^u_{ij}$$
$$= c^1(S) + c^1(T) + c^u(S) + c^u(T)$$

Theorem 4

$c^1(N) + c^u(N) = 1$.

Proof .

$$c^1(N) + c^u(N) = \sum_{i=1}^{m} w_i \sum_{j=1}^{n} x^1_{ij} + \sum_{i=1}^{m} w_i \sum_{j=1}^{n} x^u_{ij} = \sum_{i=1}^{m} w_i \left(\sum_{j=1}^{n} (x^1_{ij} + x^u_{ij}) \right) = \sum_{i=1}^{m} w_i = 1$$

A DEA minimum game

The opposite side of the game can be constructed by (*N,d*) as follows :

$$\begin{aligned} d^1(k) &= Min \sum_{i=1}^{m} w_i^k x^1_{ik} \\ s.t \ & \sum_{i=1}^{m} w_i^k = 1 \\ & w_i^k \geq 0 \quad \forall i \end{aligned} \tag{12}$$

$$\begin{aligned} d^u(k) &= Min \sum_{i=1}^{m} w_i^k x^u_{ik} \\ s.t \ & \sum_{i=1}^{m} w_i^k = 1 \\ & w_i^k \geq 0 \quad \forall i \end{aligned} \tag{13}$$

The optimal value $d^1(k) + d^u(k)$ assures the minimum division that player *k* can expect from the game

Theorem 5

$$\sum_{k=1}^{n} d^1(k) + d^u(k) \leq 1 \tag{14}$$

Analogously to the max game, for the coalition $S \subset N$, we define

$$\begin{aligned} d^1(S) &= Min \sum_{i=1}^{m} w_i x^1_i(S) \\ s.t \ & \sum_{i=1}^{m} w_i = 1 \\ & w_i \geq 0 \quad \forall i \end{aligned} \tag{15}$$

$$\begin{aligned} d^u(S) &= Min \sum_{i=1}^{m} w_i x^u_i(S) \\ s.t \ & \sum_{i=1}^{m} w_i = 1 \\ & w_i \geq 0 \quad \forall i \end{aligned} \tag{16}$$

Theorem 6

The min game *(N,d)* is super – additive we have $d^1(S \cup T) + d^u(S \cup T) \geq d^1(S) + d^1(T) + d^u(S) + d^u(T)$ for each *S, T* $\subset N$ with $S \cap T = \phi$

Proof

By renumbering the indexes, we have $S = \{1,...,h\}, T = \{h+1,...,k\}$ and $S \cup T = \{1,...,k\}$. For these sets it holds that

$$d^1(S \cup T) + d^u(S \cup T)$$
$$= Min \sum_{j=1}^{k} x^1_{ij} + Min \sum_{j=1}^{k} x^u_{ij} \geq Min \sum_{j=1}^{h} x^1_{ij}$$
$$+ Min \sum_{j=h+1}^{k} x^1_{ij} + Min \sum_{j=1}^{h} x^u_{ij} + Min \sum_{j=h+1}^{k} x^u_{ij}$$
$$= d^1(S) + d^1(T) + d^u(S) + d^u(T)$$

Thus these game start from $d^1(k) > 0$ and $d^u(k) > 0$ and enlarges the gains by the coalition until the grand coalition *N* with $c^1(N) + c^u(N) = 1$ is reached.

Theorem 7

$d^1(S) + d^u(S) + c^1(N \setminus S) + c^u(N \setminus S) = 1 \forall S \underset{\neq}{\subset} N$

Proof

By renumbering the indexes, we can assume that $S = \{1,...,h\}, N = \{1,...,n\}$ and $N \setminus S = \{h+1,...,n\}$ .For this sets, it holds that

$$d^1(S) + d^u(S) + c^1(N \setminus S) + c^u(N \setminus S)$$
$$= \underset{i}{Min} \sum_{j=1}^{h} x^1_{ij} + \underset{i}{Min} \sum_{j=1}^{h} x^u_{ij}$$
$$+ \underset{i}{Max} \sum_{j=h+1}^{n} x^1_{ij} + \underset{i}{Max} \sum_{j=h+1}^{n} x^u_{ij}$$

$$= \underset{i}{Min} \left(\sum_{j=1}^{n} x_{ij}^{1} - \sum_{j=h+1}^{n} x_{ij}^{1}\right)$$

$$+ \underset{i}{Min} \left(\sum_{j=1}^{n} x_{ij}^{u} - \sum_{\substack{j=h+1 \\ i}}^{n} x_{ij}^{u}\right) + Max \sum_{j=h+1}^{n} x_{ij}^{1}$$

$$+ Max \sum_{j=h+1}^{n} x_{ij}^{u} = \underset{i}{Min} \left(\sum_{j=h+1}^{n} x_{ij}^{1} - \sum_{j=h+1}^{n} x_{ij}^{u}\right)$$

$$- \underset{i}{Min} \left(\sum_{j=h+1}^{n} x_{ij}^{1} + \sum_{j=h+1}^{n} x_{ij}^{u}\right)$$

$$+ \underset{i}{Max} \sum_{j=h+1}^{n} x_{ij}^{1} + \underset{i}{Max} \sum_{j=h+1}^{n} x_{ij}^{u} = 1$$

EXTENTIONS

In this section, we extend basic model to maximal allocated benefit and minimal allocated cost for interval data and discuss the zero weight.

Maximal allocated benefit

Suppose that there are *s* criteria for representing benefits. Let $[y_{ij}^{1}, y_{ij}^{u}](i = 1,...,s)$ be the benefits of player $j\,(j = 1,...,n)$ where ***u*** $(u_1,...,u_s)$ is the virtual weights for benefits. Analogous to the expression (1) we define the relative score of player *j* to the total scores as:

$$\frac{\sum_{i=1}^{s} u_i y_{ij}^{1}}{\sum_{i=1}^{s} u_i \left(\sum_{j=1}^{n} (y_{ij}^{1} + y_{ij}^{u})\right)} \qquad (17)$$

$$\frac{\sum_{i=1}^{s} u_i y_{ij}^{u}}{\sum_{i=1}^{s} u_i \left(\sum_{j=1}^{n} (y_{ij}^{1} + y_{ij}^{u})\right)} \qquad (18)$$

Player *j* wishes to maximize his benefits. We can express this situation by linear programs below:

$$\begin{aligned} & Max \sum_{i=1}^{s} u_i \; y_{ij}^{1} \\ & s.t \sum_{i=1}^{s} u_i \left(\sum_{j=1}^{n} y_{ij}^{1}\right) = 1 \\ & \sum_{i=1}^{s} u_i \; y_{ij}^{1} \geq 0 \quad (j = 1,...,n) \\ & u_i \geq 0 \quad \forall i \end{aligned} \qquad (19)$$

$$\begin{aligned} & Max \sum_{i=1}^{s} u_i \; y_{ij}^{u} \\ & s.t \sum_{i=1}^{s} u_i \left(\sum_{j=1}^{n} y_{ij}^{u}\right) = 1 \\ & \sum_{i=1}^{s} u_i \; y_{ij}^{u} \geq 0 \quad (j = 1,...,n) \\ & u_i \geq 0 \quad \forall i \end{aligned} \qquad (20)$$

The weights of benefits are nonnegative. A characteristic function of the coalition *S* is defined by the linear program below:

$$\begin{aligned} & c^{1}(S) = Max \sum_{i=1}^{s} u_i \sum_{j \in S} y_{ij}^{1} \\ & s.t \sum_{i=1}^{s} u_i \left(\sum_{j=1}^{n} y_{ij}^{1}\right) = 1 \\ & \sum_{i=1}^{s} u_i \; y_{ij}^{1} \geq 0 \quad (j = 1,...,n) \\ & u_i \geq 0 \quad \forall i \end{aligned} \qquad (21)$$

$$\begin{aligned} & c^{u}(S) = Max \sum_{i=1}^{s} u_i \sum_{j \in S} y_{ij}^{u} \\ & s.t \sum_{i=1}^{s} u_i \left(\sum_{j=1}^{n} y_{ij}^{u}\right) = 1 \\ & \sum_{i=1}^{s} u_i \; y_{ij}^{u} \geq 0 \quad (j = 1,...,n) \\ & u_i \geq 0 \quad \forall i \end{aligned} \qquad (22)$$

In the program (18), (19) the benefits of all players are nonnegative. Since the constraints of program (18) and (19) are the same for all coalitions, we have the following theorem.

Theorem 8

The maximal allocated benefits game satisfies a sub-additive property.

Proof

For any $S \subset N$ and $T \subset N$ with $S \cap T = \phi$, we have:

$$c^1(S\cup T)+c^u(S\cup T)$$
$$=Max\sum_{i=1}^{s}u_i\sum_{j\in S\cup T}y^1_{ij}+Max\sum_{i=1}^{s}u_i\sum_{j\in S\cup T}y^u_{ij}$$
$$=Max\left(\sum_{i=1}^{s}u_i(\sum_{j\in S}y^1_{ij}+\sum_{j\in T}y^1_{ij})\right)$$
$$+Max\left(\sum_{i=1}^{s}u_i(\sum_{j\in S}y^u_{ij}+\sum_{j\in T}y^u_{ij})\right)$$
$$=Max\left((\sum_{i=1}^{s}u_i\sum_{j\in S}y^1_{ij})+(\sum_{i=1}^{s}u_i\sum_{j\in T}y^1_{ij})\right)$$
$$+Max\left((\sum_{i=1}^{s}u_i\sum_{j\in S}y^u_{ij})+(\sum_{i=1}^{s}u_i\sum_{j\in T}y^u_{ij})\right)$$
$$\le Max\sum_{i=1}^{s}u_i\sum_{j\in S}y^1_{ij}+Max\sum_{i=1}^{s}u_i\sum_{j\in T}y^1_{ij}$$
$$+Max\sum_{i=1}^{s}u_i\sum_{j\in S}y^u_{ij}+Max\sum_{i=1}^{s}u_i\sum_{j\in T}y^u_{ij}$$
$$\le c^1(S)+c^u(S)+c^1(T)+c^u(T)$$

Minimal allocated cost

Suppose that there are *m* criteria for representing costs. Let $[x^1_{ij},x^u_{ij}](i=1,...,m)$ be the costs of player $j\,(j=1,...,n)$ where ***v*** $(v_1,...,v_m)$ is the virtual weights for costs. player *j* wishes to minimize his costs then we have:

$$\begin{aligned}&Min\ \sum_{i=1}^{m}v_ix^1_{ij}\\&s.t\quad \sum_{i=1}^{m}v_i\sum_{j=1}^{n}x^1_{ij}=1\\&\qquad \sum_{i=1}^{m}v_ix^1_{ij}\ge 0\quad (j=1,...,n)\\&\qquad v_i\ge 0\qquad \forall i\end{aligned}\tag{23}$$

$$\begin{aligned}&Min\ \sum_{i=1}^{m}v_ix^u_{ij}\\&s.t\quad \sum_{i=1}^{m}v_i\sum_{j=1}^{n}x^u_{ij}=1\\&\qquad \sum_{i=1}^{m}v_ix^u_{ij}\ge 0\quad (j=1,...,n)\\&\qquad v_i\ge 0\qquad \forall i\end{aligned}\tag{24}$$

The weights of costs are nonnegative. A characteristic function of the coalition S is defined by the linear program below

$$\begin{aligned}&d(S)=Min\ \sum_{i=1}^{m}v_i\sum_{j\in S}x^1_{ij}\\&s.t\quad \sum_{i=1}^{m}v_i\sum_{j=1}^{n}x^1_{ij}=1\\&\qquad \sum_{i=1}^{m}v_ix^1_{ij}\ge 0\quad (j=1,...,n)\\&\qquad v_i\ge 0\qquad \forall i\end{aligned}\tag{25}$$

$$\begin{aligned}&d(S)=Min\ \sum_{i=1}^{m}v_i\sum_{j\in S}x^u_{ij}\\&s.t\quad \sum_{i=1}^{m}v_i\sum_{j=1}^{n}x^u_{ij}=1\\&\qquad \sum_{i=1}^{m}v_ix^u_{ij}\ge 0\quad (j=1,...,n)\\&\qquad v_i\ge 0\qquad \forall i\end{aligned}\tag{26}$$

In the program (22), (23) the costs of all players are nonnegative. Minimal allocated costs game, satisfies a super –additive property.

Theorem 9

The maximal allocated cost game (*N,c*) and min game (*N,d*) are dual games, for any S ⊂ N, we have $d^1(S)+d^u(S)+c^1(N\setminus S)+c^u(N\setminus S)=1$.

Proof

$$c^1(N\setminus S)+c^u(N\setminus S)=$$
$$\underset{u}{Max}(\sum_{i=1}^{s}u_i\sum_{j\in N\setminus S}y^1_{ij})+\underset{u}{Max}(\sum_{i=1}^{s}u_i\sum_{j\in N\setminus S}y^u_{ij})$$
$$=\underset{u}{Max}(\sum_{i=1}^{s}u_i(\sum_{j\in N}y^1_{ij}-\sum_{j\in S}y^1_{ij})+$$
$$\underset{u}{Max}(\sum_{i=1}^{s}u_i(\sum_{j\in N}y^u_{ij}-\sum_{j\in S}y^u_{ij})$$
$$=\underset{u}{Max}(\sum_{i=1}^{s}u_i\sum_{j\in N}y^1_{ij}-\sum_{i=1}^{s}u_i\sum_{j\in S}y^1_{ij})$$
$$+\underset{u}{Max}(\sum_{i=1}^{s}u_i\sum_{j\in N}y^u_{ij}-\sum_{i=1}^{s}u_i\sum_{j\in S}y^u_{ij})$$
$$=\underset{u}{Max}(1-\sum_{i=1}^{s}u_i\sum_{j\in S}y^1_{ij})+\underset{u}{Max}(1-\sum_{i=1}^{s}u_i\sum_{j\in S}y^u_{ij})$$
$$=1-\underset{u}{Min}(\sum_{i=1}^{s}u_i\sum_{j\in S}y^1_{ij})-\underset{u}{Min}(\sum_{i=1}^{s}u_i\sum_{j\in S}y^u_{ij})$$
$$=1-d^1(S)-d^u(S).$$

Avoiding occurrence of zero weights and setting preference on weights

In previous sections, we presented a scheme for determining the weights through the program (18), (19), (22) and (23). Some weight may happen to be zero for all optimal solutions. This means that the corresponding criterion is not accounted for in the solution of the game at all. Let us suppose that all players agree to put preference on certain criteria. The zero weight issue can thus be solved in this way (Allen et al., 1997). If all players agree to incorporate preference regarding criteria, we can apply the following "assurance region method". For example , we set constraints on the ratio w_1, w_i *(i=2,…,m)* as: $L_i \le w_i / w_1 \le U_i$, *(i*=2,…,*m*) where L_i and U_i denote lower and upper bounds of the ratio w_i / w_1 , respectively. These bounds must be set by agreement among all players. The program (5) is now modified as:

$$
\begin{aligned}
c(S) = Max\ & \sum_{i=1}^{m} w_i x_i^1(s) \\
s.t\ & \sum_{i=1}^{m} w_i = 1 \\
& L_i \le \frac{w_i}{w_1} \le U_i \quad (i = 2,\dots,m) \\
& w_i \ge 0 \quad \forall i
\end{aligned} \tag{27}
$$

Similarly, we can avoid occurrence of zero weight in linear programs of maximal allocated benefits and minimal allocated costs. Then, we have:

$$
\begin{aligned}
c^1(S) = Max & \sum_{i=1}^{s} u_i \sum_{j\in S} y_{ij}^1 \\
s.t & \sum_{i=1}^{s} u_i \sum_{j=1}^{n} y_{ij}^1 = 1 \\
& \sum_{i=1}^{s} u_i y_{ij}^1 \ge 0 \quad (i = 1,\dots,n) \\
& L_i \le \frac{u_i}{u_1} \le U_i \ (i = 2,\dots,s) \\
& u_i \ge 0 \quad \forall i
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
c^u(S) = Max & \sum_{i=1}^{s} u_i \sum_{j\in S} y_{ij}^u \\
s.t & \sum_{i=1}^{s} u_i \sum_{j=1}^{n} y_{ij}^u = 1 \\
& \sum_{i=1}^{s} u_i y_{ij}^u \ge 0 \quad (i = 1,\dots,n) \\
& L_i \le \frac{u_i}{u_1} \le U_i \ (i = 2,\dots,s) \\
& u_i \ge 0 \quad \forall i
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
d^1(S) = Min & \sum_{i=1}^{m} v_i \sum_{j\in S} x_{ij}^1 \\
s.t & \sum_{i=1}^{m} v_i \sum_{j=1}^{n} x_{ij}^1 = 1 \\
& \sum_{i=1}^{m} v_i x_{ij}^1 \ge 0 \quad (i = 1,\dots,n) \\
& L_i \le \frac{v_i}{v_1} \le U_i \ (i = 2,\dots,m) \\
& v_i \ge 0 \quad \forall i
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
d^u(S) = Min & \sum_{i=1}^{m} v_i \sum_{j\in S} x_{ij}^u \\
s.t & \sum_{i=1}^{m} v_i \sum_{j=1}^{n} x_{ij}^u = 1 \\
& \sum_{i=1}^{m} v_i x_{ij}^u \ge 0 \quad (i = 1,\dots,n) \\
& L_i \le \frac{v_i}{v_1} \le U_i \ (i = 2,\dots,m) \\
& v_i \ge 0 \quad \forall i
\end{aligned} \tag{31}
$$

The (22), (23) are modified in (27), (28) respectively.

The best coalition

In program (18), (19),(22) and (23) we presented a scheme for computing maximal allocated benefits, minimal allocated cost for coalitions. Also we can compute maximal allocated benefits and minimal allocated costs for members of coalition, using programs (16),(17),(20) and (21).These values can determine the players expected percentages of the total benefit, cost in the game. Each player can increase benefit allocation and decrease cost allocation, establishing coalition. In other words the possible ultimate benefit allocated to the coalition can increase and minimal cost allocated can decrease once the best circumstance is provided. There is a question, how can player j establish coalition? Now, knowing this it's easy for the player to examine which other players, she/he can establish coalition with so that he can reach the ultimate benefit ratio and minimal cost ratio. Each player can establish coalition in different ways, chance coalition, coalition concerning players of minimal allocated benefit (cost), coalition concerning players of maximal allocated benefit (cost), coalition with the player enjoying the ultimate benefit and the players with the minimal allocated benefit (cost), coalition with the player enjoying the minimal benefit (cost) and the players with maximal allocated benefit (cost). Clearly, coalition with the player having the ultimate allocated benefit would be better than the others. Coalition with the player having the minimal allocated benefit would be poorer in comparison to others. Now, it's easy to understand that a player with the minimal benefit ratio establish a coalition with the player who has allocated the ultimate benefit for him/her self and player with the maximal benefit ratio establish a coalition with the player who has allocated the minimal benefit for him/her self. Having established the coalition the player cost ratio would be less or unchanged. Clearly, player with the minimal cost ratio establish a coalition with the player who has allocated the minimal cost for him/herself, and player with the maximal cost establish a coalition with the player who has allocated the minimal cost for himself. These results represented by the example below.

Table 1. Cost and benefit criteria for first of interval.

Player j	x^1_{1j}	x^1_{2j}	y^1_{1j}	y^1_{2j}	y^1_{3j}	y^1_{4j}
1	50	20	800	200	350	340
2	70	18	900	160	320	470
3	80	22	1000	175	395	400
4	110	30	950	185	290	510
5	90	17	960	186	280	480
6	55	24	870	210	360	370
7	65	26	780	165	300	440
8	75	32	670	150	400	500
9	50	29	810	170	410	510
10	100	16	910	190	420	390

Table 2. Maximal allocated cost of obtained total game benefit.

Player *j*	Maximal allocated benefit for first of interval (%)	Minimal allocated cost for first of interval (%)
1	11.17	6.71
2	10.66	7.69
3	11.56	9.40
4	11.56	12.82
5	11.10	7.26
6	11.73	7.3
7	9.98	8.72
8	11.35	10.7
9	11.63	6.71
10	11.91	6.84

NUMERICAL RESULTS

There are 10 players in this game. Each player uses 2 cost criteria and 4 benefit criteria for first of interval (Table1). We compute maximal allocated benefit and the minimal allocated cost for each player for first of interval. Table 2 show this results (we set constraints in the ratio u_1). Similarly, we can compute maximal allocated benefit and minimal allocated cost for end of interval. We now apply this approach to the data in Table 2. 7th player has the minimal allocated benefit and the l0th has the maximal allocated benefit. Also, the 1st has maximal allocated cost and 4th and 9th have the minimal allocated cost.

Consider in Table 3 (A) , S_1 is chance coalition , S_2 is coalition concerning players of minimal allocated benefit, S_3 is coalition concerning players of maximal allocated benefit, S_4 is coalition with the player enjoying the minimal benefit and the players with maximal allocated benefit and in Table 3 (B), S_6 is chance coalition S_7 is coalition concerning players of minimal allocated cost, S_8 is concerning players of maximal allocated, S_9 is coalition with the players enjoying the minimal cost and the players with maximal allocated cost and S_{10} is coalition with the player enjoying the maximal cost and the players with minimal allocated cost.

Table 4 shows, establish the coalition the player benefit ratio are increased or unchanged. Also, establish the coalition the player cost ratio are decreased or unchanged. 7th player with the least benefit ratio has the most benefit inS_4.10th player with the maximal benefit ratio has the most benefit in S_5.4th player with the most cost ratio has the least cost inS_{10}.1st player with the minimal cost ratio has the least cost in S_7.

Conclusion

In this paper, we have studied the common weight issues that connect the game solution with arbitrary weight selection behavior of the players. Regarding this subject, we have proposed a method for compute maximal allocated benefit and minimal allocated costs for interval data. We have introduced coalitions and the ways for finding the best coalitions. In this sense, avoided occurrence zero weight by assurance region method.

Table 3(a). The coalitions and maximal allocated benefit for them and their members.

Coalition	Maximal allocated benefits for coalition	Members of coalition	Maximal allocated benefits of total game benefits (%)
		10	12.25
S_1	31.77	8	11.94
		7	10.51
		10	11.92
S_2	33.76	6	12.44
		9	13.56
		5	11.47
S_3	31.52	2	11.21
		7	10.51
		7	11.47
S_4	31.55	10	12.27
		6	11.73
		10	12.09
S_5	29.94	2	10.12
		7	10.13

Table 3(b). The coalitions and minimal allocated cost for them and their members.

Coalition	Minimal allocated cost for coalition	Members of coalition	Minimal allocated cost of total game cost (%)
		1	6.71
S_1	28.86	4	11.69
		6	7.3
S_2	13.42	1	5.47
		9	6.71
		4	12.55
S_3	35.17	8	10.06
		3	9.21
		4	11.54
S_4	31.54	8	10.06
		1	6.71
		4	10.70
S_5	28.19	9	6.71
		1	6.71

Table 4. Modified allocated benefits and costs (We set constraints in the ratio U_1,V_1)

Player *j*	Modified maximal allocated benefit for first of interval (%)	Modified minimal allocated cost for first of interval (%)
1	9.22	6.80
2	10.33	9.13
3	11.12	10.53
4	10.90	14.47
5	10.78	11.34
6	9.91	7.52
7	9.22	8.84
8	9.11	10.24
9	10.20	6.98
10	10.52	12.41

Furthermore a numerical example, have been calculated with proposed ways, has been considered.

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