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Discretization and Analysis of HIV-1 and HTLV-I Coinfection Model with Latent Reservoirs

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Abstract: This article formulates and analyzes a discrete-time Human immunodeficiency virus type 1 (HIV-1) and human T-lymphotropic virus type I (HTLV-I) coinfection model with latent reservoirs. We consider that the HTLV-I infect the $CD4^+T$ cells, while HIV-1 has two classes of target cells—$CD4^+T$ cells and macrophages. The discrete-time model is obtained by discretizing the original continuous-time by the non-standard finite difference (NSFD) approach. We establish that NSFD maintains the positivity and boundedness of the model's solutions. We derived four threshold parameters that determine the existence and stability of the four equilibria of the model. The Lyapunov method is used to examine the global stability of all equilibria. The analytical findings are supported via numerical simulation. The impact of latent reservoirs on the HIV-1 and HTLV-I co-dynamics is discussed. We show that incorporating the latent reservoirs into the HIV-1 and HTLV-I coinfection model will reduce the basic HIV-1 single-infection and HTLV-I single-infection reproductive numbers. We establish that neglecting the latent reservoirs will lead to overestimation of the required HIV-1 antiviral drugs. Moreover, we show that lengthening of the latent phase can suppress the progression of viral coinfection. This may draw the attention of scientists and pharmaceutical companies to create new treatments that prolong the latency period.

Keywords: non-standard finite difference; viral coinfection; Lyapunov stability; latency

MSC: 37M05; 39A10; 65L12; 92D25; 92D30

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1. Introduction

Over the past decades, many scientists and researchers from the disciplines of mathematics, biology, and medicine have been interested in modeling the dynamics of viral infection within the host. Mathematical modeling of viral infection has a long history of helping to provide insight that is difficult to obtain through pure experiments. Examples of viral single-infection that have been modeled and studied are as follows: (i) chronic viral infections such as human immunodeficiency virus type 1 (HIV-1) [1], human T-lymphotropic virus type I (HTLV-I) [2], hepatitis B virus (HBV) [3] and hepatitis C virus (HCV) [4], (ii) respiratory viral infections such as influenza A virus (IAV) [5] and severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) [6,7], (iii) vector-borne viral infections such as dengue virus [8], chikungunya virus [9] and Zika virus [10]. The in-host viral coinfections were also modeled in recent years such as Zika/dengue [11], HIV-1/HTLV-I [12,13], IAV/SARS-CoV-2 [14,15], SARS-CoV-2/HIV-1 [16], SARS-CoV-2/HTLV-I [17], HIV-1/HCV [18] and HIV-1/HBV [19]. HIV-1 and HTLV-I are two dangerous retroviruses that attack the central component of the immune system, $CD4^+T$ cells and can cause chronic diseases. HIV-1 causes acquired immunodeficiency syndrome (AIDS), while HTLV-I causes HTLV-I-associated myelopathy/tropical spastic paraparesis (HAM/TSP) and adult T-cell leukemia (ATL) diseases. Cytotoxic T lymphocytes (CTLs) and B cells play important roles in the immune response against viral infection. CTLs attack and kill the viral-infected cells, while B cells produce antibodies to neutralize viruses.

Since formulating the basic model of HIV-1 single-infection in [1], many modifications and developments have been made to make the model more accurate in describing the dynamics of the virus within the host. Some biological factors have been taken into account such as: (i) time delay [20,21], (ii) drug therapies [20,22], (iii) CTL immunity [1,23], (iv) antibody immunity [24], (v) reaction–diffusion [25], (vi) stochastic effects [26], and (vii) two target cells ($CD4^{+}T$ cells and macrophages) [22,27–29].

Latent HIV-1 reservoirs are considered a significant obstacle to HIV-1 elimination [30]. Latent HIV-1-infected cells can be activated after a period of time and then become viral production cells when drug therapies are stopped. These cells contains the HIV-1 virions; however, they do not produce them until they are stimulated. This fact causes an escapement of latent HIV-1-infected cells from the immune response. HIV-1 dynamics models with latent infected cells were developed in several works (see e.g., [23,26,30–32].

Stilianakis and Seydel [2] formulated a mathematical model for in-host HTLV-I dynamics. After that, several HTLV-I dynamics models were developed. HTLV-I infection models with CTL immune response were addressed in [33–36]. HTLV-I infection models have been incorporated with intracellular delay in [37], and with immune response delay in [35,37]. Reaction-diffusion HTLV-I infection models were investigated in [34].

HIV-1 and HTLV-1 share the methods of transmission between people through sexual relationships, infected sharp objects, blood transfusions and organ transplantation. Therefore, some nonlinear continuous-time models were recently formulated that describe the dynamics of HIV-1 and HTLV-1 coinfection in-host [12,13]. In [12,13], it was assumed that HIV-1 has one type of target cells, $CD4^{+}T$ cells. In fact, HIV-1 can also infect macrophages [31]. In [38], an HIV-1 and HTLV-I coinfection model was formulated by considering two types of target cells for HIV-1, $CD4^{+}T$ cells and macrophages:

$$\begin{cases} \frac{dx}{dt} = \overbrace{\zeta_1}^{\text{CD4}^+\text{T cells production}} - \overbrace{\varrho_1 x}^{\text{death}} - \overbrace{\rho_1 x v}^{\text{HIV-1 infectious transmission}} - \overbrace{\rho_3 x u}^{\text{HTLV-I infectious transmission}}, \\ \frac{dy}{dt} = \overbrace{\rho_1 x v}^{\text{HIV-1 infectious transmission}} - \overbrace{\alpha_1 y}^{\text{death}}, \\ \frac{dw}{dt} = \overbrace{\zeta_2}^{\text{macrophages production}} - \overbrace{\varrho_2 w}^{\text{death}} - \overbrace{\rho_2 w v}^{\text{HIV-1 infectious transmission}}, \\ \frac{dz}{dt} = \overbrace{\rho_2 w v}^{\text{HIV-1 infectious transmission}} - \overbrace{\alpha_2 z}^{\text{death}}, \\ \frac{dv}{dt} = \overbrace{\beta_1 \alpha_1 y + \beta_2 \alpha_2 z}^{\text{generation of HIV-1}} - \overbrace{\theta v}^{\text{death}}, \\ \frac{du}{dt} = \overbrace{\rho_3 x u}^{\text{HTLV-I infectious transmission}} - \overbrace{\delta u}^{\text{death}}, \end{cases} \tag{1}$$

where x, y, w, z, v and u denote the concentrations of uninfected $CD4^{+}T$ cells, HIV-1-infected $CD4^{+}T$ cells, uninfected macrophages, HIV-1-infected macrophages, HIV-1 particles and HTLV-I-infected $CD4^{+}T$ cells, respectively. Model (1) was discretized by the NSFD method in [39]. Global stability of the discretized model was established using the Lyapunov technique.

In model (1), latent HIV-1 and HTLV-I reservoirs were not included. Therefore, model (1) has been extended in [40] by including three additional populations, latent

HIV-1-infected CD4$^+$T cells (g), latent HIV-1-infected macrophages (s) and latent HTLV-I-infected CD4$^+$T cells (q) as:

$$\begin{cases} \frac{dx}{dt} = \zeta_1 - \varrho_1 x - \rho_1 x v - \rho_3 x u, \\ \frac{dg}{dt} = \rho_1 x v - (\pi_1 + \mu_1) g, \\ \frac{dy}{dt} = \pi_1 g - \alpha_1 y, \\ \frac{dw}{dt} = \zeta_2 - \varrho_2 w - \rho_2 w v, \\ \frac{ds}{dt} = \rho_2 w v - (\pi_2 + \mu_2) s, \\ \frac{dz}{dt} = \pi_2 s - \alpha_2 z, \\ \frac{dv}{dt} = \beta_1 \alpha_1 y + \beta_2 \alpha_2 z - \theta v, \\ \frac{dq}{dt} = \rho_3 x u - (\pi_3 + \mu_3) q, \\ \frac{du}{dt} = \pi_3 q - \delta u, \end{cases} \tag{2}$$

where the latent HIV-1-infected CD4$^+$T cells, latent HIV-1-infected macrophages and latent HTLV-I-infected CD4$^+$T cells are activated by rates $\pi_1 g$, $\pi_2 s$ and $\pi_3 q$, respectively, while they die at rates $\mu_1 g$, $\mu_2 s$ and $\mu_3 q$, respectively.

We note that model (2) is nonlinear continuous-time, and its exact analytical solution is unknown; therefore, discretization is unavoidable. Further, blood measurements from infected patients can only be available at discrete-time instants. As a result, an adequate discretization approach has to be chosen such that the basic and global properties of the original model is maintained. Mickens [41] introduced a non-standard finite difference (NSFD) scheme for solving different types of differential equations. NSFD was successfully utilized in discretizing several within-host virus dynamics models [42–45]. HIV-1 continuous-time models were discretized via the NSFD approach in [46–49]. A stability analysis of a discrete HIV-1 dynamics model with the Beddington–DeAngelis incidence and cure rate was studied in [47]. In [49], the global stability of discrete HIV-1 dynamics models with three classes of HIV-1-infected cells was studied. In [48], the HIV-1 dynamics model given by PDEs was discretized via the NSFD method. The Lyapunov method was used to prove the global stability of equilibria.

We mention that all mathematical models for HTLV-I single-infection and HIV-1/HTLV-I coinfection presented in the literature are given as continuous-time systems. The only exception is our recent paper [39], where a discrete HIV-1 and HTLV-I coinfection model is considered. In [39], the presence of latent reservoirs has not been modeled. The aim of the present article is to use the NSFD method to discretize an HIV-1 and HTLV-I coinfection model with latent reservoirs. We first establish the positivity and ultimate boundedness of the discrete-time model's solutions, then calculate all equilibria and deduce their existence conditions. We examine the global stability of the four equilibria using the Lyapunov approach. We present some numerical simulations to clarify the theoretical results. We discuss the impact of latent reservoirs on the HIV-1 and HTLV-I co-dynamics.

2. The Discrete-Time Model

Classical numerical methods, such as Euler, Runge–Kutta and others, when used in solving nonlinear differential equations, suffer from numerical instability and bias when large step sizes are used in the numerical simulation [50]. In this situation, these numerical methods may provide non-physical solutions and can produce 'false' or 'spurious' fixed points, which are not fixed points of the original continuous-time model [51,52]. The NSFD method preserves the essential qualitative features of the original continuous-time model such as equilibria, positivity, boundedness and global behaviors of solutions independently of the selected step-size.

Applying the NSFD approach on system (2), we obtain

$$\frac{x_{n+1}-x_n}{\Upsilon(h)} = \zeta_1 - \varrho_1 x_{n+1} - \rho_1 x_{n+1} v_n - \rho_3 x_{n+1} u_n, \tag{3}$$

$$\frac{g_{n+1}-g_n}{\Upsilon(h)} = \rho_1 x_{n+1} v_n - (\pi_1 + \mu_1) g_{n+1}, \tag{4}$$

$$\frac{y_{n+1}-y_n}{\Upsilon(h)} = \pi_1 g_{n+1} - \alpha_1 y_{n+1}, \tag{5}$$

$$\frac{w_{n+1}-w_n}{\Upsilon(h)} = \zeta_2 - \varrho_2 w_{n+1} - \rho_2 w_{n+1} v_n, \tag{6}$$

$$\frac{s_{n+1}-s_n}{\Upsilon(h)} = \rho_2 w_{n+1} v_n - (\pi_2 + \mu_2) s_{n+1}, \tag{7}$$

$$\frac{z_{n+1}-z_n}{\Upsilon(h)} = \pi_2 s_{n+1} - \alpha_2 z_{n+1}, \tag{8}$$

$$\frac{v_{n+1}-v_n}{\Upsilon(h)} = \beta_1 \alpha_1 y_{n+1} + \beta_2 \alpha_2 z_{n+1} - \theta v_{n+1}, \tag{9}$$

$$\frac{q_{n+1}-q_n}{\Upsilon(h)} = \rho_3 x_{n+1} u_n - (\pi_3 + \mu_3) q_{n+1}, \tag{10}$$

$$\frac{u_{n+1}-u_n}{\Upsilon(h)} = \pi_3 q_{n+1} - \delta u_{n+1}, \tag{11}$$

where $h > 0$ is the time step, and $(x_n, g_n, y_n, w_n, s_n, z_n, v_n, q_n, u_n)$ are the approximation of the solution $(x(t_n), g(t_n), y(t_n), w(t_n), s(t_n), z(t_n), v(t_n), q(t_n), u(t_n))$ of the system (2) at the discrete time point $t_n = nh$, $n \in N = \{0, 1, 2, \ldots\}$. The denominator function $\Upsilon(h)$ is selected such that $\Upsilon(h) = h + O(h^2)$. We consider the following form of $\Upsilon(h)$

$$\Upsilon(h) = \frac{1 - e^{-\varrho_1 h}}{\varrho_1}. \tag{12}$$

The initial conditions of system (3)–(11) are

$$(x_0, g_0, y_0, w_0, s_0, z_0, v_0, q_0, u_0) \in \mathbb{R}^9_+ = \{(x, g, y, w, s, z, v, q, u) \mid x > 0, g > 0, y > 0, w > 0, s > 0, z > 0, v > 0, q > 0, u > 0\}. \tag{13}$$

3. Preliminaries

Let $\sigma = \min\{\varrho_1, \alpha_1, \delta, \varrho_2, \alpha_2, \mu_1, \mu_2, \mu_3\}$, and $\zeta_{12} = \zeta_1 + \zeta_2$ and define the regions

$$\Gamma_1 = \Big\{(x, g, y, w, s, z, v, q, u) \in \mathbb{R}^9_+ : x \le \frac{\zeta_1}{\varrho_1}, w \le \frac{\zeta_2}{\varrho_2}$$
$$0 < x + g + y + w + s + z + q + u \le \frac{\zeta_{12}}{\sigma}, v \le \frac{(\beta_1\alpha_1 + \beta_2\alpha_2)\zeta_{12}}{\theta\sigma}\Big\},$$
$$\Gamma_0 = \Big\{(x, 0, 0, w, 0, 0, 0, 0, 0) \in \mathbb{R}^9_+ : x \ge 0, w \ge 0\Big\}.$$

Lemma 1. *Any solution $(x, g, y, w, s, z, v, q, u)$ of model (3)–(11) with initial conditions (13) is positive and ultimately bounded.*

Proof. Equations (3)–(11) imply that

$$x_{n+1} = \frac{\Upsilon(h)\zeta_1 + x_n}{1 + \Upsilon(h)(\varrho_1 + \rho_1 v_n + \rho_3 u_n)}, \tag{14}$$

$$g_{n+1} = \frac{\Upsilon(h)\rho_1 x_{n+1} v_n + g_n}{1 + \Upsilon(h)(\pi_1 + \mu_1)}, \tag{15}$$

$$y_{n+1} = \frac{\Upsilon(h)\pi_1 g_{n+1} + y_n}{1 + \Upsilon(h)\alpha_1}, \tag{16}$$

$$w_{n+1} = \frac{\Upsilon(h)\zeta_2 + w_n}{1 + \Upsilon(h)(\varrho_2 + \rho_2 v_n)}, \tag{17}$$

$$s_{n+1} = \frac{\Upsilon(h)\rho_2 w_{n+1} v_n + s_n}{1 + \Upsilon(h)(\pi_2 + \mu_2)}, \tag{18}$$

$$z_{n+1} = \frac{\Upsilon(h)\pi_2 s_{n+1} + z_n}{1 + \Upsilon(h)\alpha_2}, \tag{19}$$

$$v_{n+1} = \frac{\Upsilon(h)(\beta_1\alpha_1 y_{n+1} + \beta_2\alpha_2 z_{n+1}) + v_n}{1 + \Upsilon(h)\theta}, \tag{20}$$

$$q_{n+1} = \frac{\Upsilon(h)\rho_3 x_{n+1} u_n + q_n}{1 + \Upsilon(h)(\pi_3 + \mu_3)}, \tag{21}$$

$$u_{n+1} = \frac{\Upsilon(h)\pi_3 q_{n+1} + u_n}{1 + \Upsilon(h)\delta}, \tag{22}$$

Since all parameters of model (2) are positive and the initial values are also positive, then by induction we obtain $x_n > 0, g_n > 0, y_n > 0, w_n > 0, s_n > 0, z_n > 0, v_n > 0, q_n > 0$, and $u_n > 0$ for all $n \in N$.

From Equations (3) and (6), we have

$$\begin{aligned} x_{n+1} - x_n &= \Upsilon(h)[\zeta_1 - \varrho_1 x_{n+1} - \rho_1 x_{n+1} v_n - \rho_3 x_{n+1} u_n] \\ &\leq \Upsilon(h)[\zeta_1 - \varrho_1 x_{n+1}], \\ w_{n+1} - w_n &= \Upsilon(h)[\zeta_2 - \varrho_2 w_{n+1} - \rho_2 w_{n+1} v_n], \\ &\leq \Upsilon(h)[\zeta_2 - \varrho_2 w_{n+1}]. \end{aligned}$$

It follows that

$$x_{n+1} \leq \frac{x_n}{1 + \varrho_1\Upsilon(h)} + \frac{\zeta_1\Upsilon(h)}{1 + \varrho_1\Upsilon(h)} \text{ and } w_{n+1} \leq \frac{w_n}{1 + \varrho_2\Upsilon(h)} + \frac{\zeta_2\Upsilon(h)}{1 + \varrho_2\Upsilon(h)}.$$

Using Lemma 2.2 in [53], we obtain

$$\begin{aligned} x_n &\leq \left(\frac{1}{1 + \Upsilon(h)\varrho_1}\right)^n x_0 + \frac{\zeta_1}{\varrho_1}\left[1 - \left(\frac{1}{1 + \Upsilon(h)\varrho_1}\right)^n\right], \\ w_n &\leq \left(\frac{1}{1 + \Upsilon(h)\varrho_2}\right)^n w_0 + \frac{\zeta_2}{\varrho_2}\left[1 - \left(\frac{1}{1 + \Upsilon(h)\varrho_2}\right)^n\right]. \end{aligned}$$

Consequently, $\limsup_{n\to\infty} x_n \leq \frac{\zeta_1}{\varrho_1}$ and $\limsup_{n\to\infty} w_n \leq \frac{\zeta_2}{\varrho_2}$. Define a sequence K_n as:

$$K_n = x_n + g_n + y_n + w_n + s_n + z_n + q_n + u_n.$$

Hence

$$\begin{aligned}K_{n+1}-K_n &= (x_{n+1}-x_n)+(g_{n+1}-g_n)+(y_{n+1}-y_n)+(w_{n+1}-w_n)+(s_{n+1}-s_n)+(z_{n+1}-z_n)\\ &\quad+(q_{n+1}-q_n)+(u_{n+1}-u_n)\\ &= \Upsilon(h)[\zeta_1-\varrho_1 x_{n+1}-\mu_1 g_{n+1}-\alpha_1 y_{n+1}+\zeta_2-\varrho_2 w_{n+1}-\mu_2 s_{n+1}-\alpha_2 z_{n+1}-\mu_3 q_{n+1}\\ &\quad-\delta u_{n+1}]\\ &\le \Upsilon(h)\zeta_{12}-\Upsilon(h)\sigma[x_{n+1}+g_{n+1}+y_{n+1}+w_{n+1}+s_{n+1}+z_{n+1}+u_{n+1}]\\ &= \Upsilon(h)\zeta_{12}-\Upsilon(h)\sigma K_{n+1}.\end{aligned}$$

Hence

$$K_{n+1}\le \frac{K_n}{1+\Upsilon(h)\sigma}+\frac{\Upsilon(h)\zeta_{12}}{1+\Upsilon(h)\sigma}.$$

Lemma 2.2 in [53] gives

$$K_n \le \left(\frac{1}{1+\Upsilon(h)\sigma}\right)^n K_0+\frac{\zeta_{12}}{\sigma}\left[1-\left(\frac{1}{1+\Upsilon(h)\sigma}\right)^n\right].$$

Then, $\limsup_{n\to\infty} K_n \le \frac{\zeta_{12}}{\sigma}$. From Equation (9), we obtain

$$\begin{aligned}v_{n+1}-v_n &= \Upsilon(h)[\beta_1\alpha_1 y_{n+1}+\beta_2\alpha_2 z_{n+1}-\theta v_{n+1}]\\ &\le \Upsilon(h)\left[\beta_1\alpha_1\frac{\zeta_{12}}{\sigma}+\beta_2\alpha_2\frac{\zeta_{12}}{\sigma}-\theta v_{n+1}\right].\end{aligned}$$

Hence

$$v_{n+1}\le \frac{v_n}{1+\Upsilon(h)\theta}+\frac{\Upsilon(h)(\beta_1\alpha_1+\beta_2\alpha_2)\zeta_{12}}{(1+\Upsilon(h)\theta)\sigma}.$$

By induction, we obtain

$$v_n \le \left(\frac{1}{1+\Upsilon(h)\theta}\right)^n v_0+\frac{(\beta_1\alpha_1+\beta_2\alpha_2)\zeta_{12}}{\theta\sigma}\left[1-\left(\frac{1}{1+\Upsilon(h)\theta}\right)^n\right].$$

Consequently, $\limsup_{n\to\infty} v_n \le \frac{(\beta_1\alpha_1+\beta_2\alpha_2)\zeta_{12}}{\theta\sigma}$. Therefore, $(x_{n,}, g_n, y_n, w_n, s_n, z_n, v_n, q_n, u_n)$ converge to Γ as $n\to\infty$. □

4. Equilibria

Here, we calculate the model's equilibria and deduce their existence conditions.

Lemma 2. *Model (3)–(11) has four equilibria that are determined by four threshold parameters $R_j>0, j=0,1,2,3$:*

(1) Infection-free equilibrium $EQ_0=(x^0,0,0,w^0,0,0,0,0,0)$, which always exists.

(2) Chronic HIV-1 single-infection equilibrium $EQ_1=(\hat{x},\hat{g},\hat{y},\hat{w},\hat{s},\hat{z},\hat{v},0,0)$ exists when $R_0=R_{01}+R_{02}>1$.

(3) Chronic HTLV-I single-infection equilibrium $EQ_2=(\tilde{x},0,0,\tilde{w},0,0,0,\tilde{q},\tilde{u})$ exists when $R_1>1$.

(4) Chronic HIV-1/HTLV-I coinfection infection equilibrium $EQ_3=(\bar{x},\bar{g},\bar{y},\bar{w},\bar{s},\bar{z},\bar{v},\bar{q},\bar{u})$ exists when $\frac{R_1}{R_{01}}>1$, $R_2>1$ and $R_3>1$.

Proof. Any equilibrium $EQ = (x, g, y, w, s, z, v, q, u)$ satisfies

$$0 = \zeta_1 - \varrho_1 x - \rho_1 x v - \rho_3 x u, \tag{23}$$
$$0 = \rho_1 x v - (\pi_1 + \mu_1) g \tag{24}$$
$$0 = \pi_1 g - \alpha_1 y \tag{25}$$
$$0 = \zeta_2 - \varrho_2 w - \rho_2 w v, \tag{26}$$
$$0 = \rho_2 w v - (\pi_2 + \mu_2) s \tag{27}$$
$$0 = \pi_2 s - \alpha_2 z, \tag{28}$$
$$0 = \beta_1 \alpha_1 y + \beta_2 \alpha_2 z - \theta v, \tag{29}$$
$$0 = \rho_3 x u - (\pi_3 + \mu_3) q \tag{30}$$
$$0 = \pi_3 q - \delta u. \tag{31}$$

From Equations (30) and (31), we obtain two options $u = 0$ and $x = \frac{(\pi_3+\mu_3)\delta}{\rho_3 \pi_3}$. First, we consider $u = 0$, then $q = 0$.

From Equations (23) and (26), we obtain

$$x = \frac{\zeta_1}{\varrho_1 + \rho_1 v}, \quad w = \frac{\zeta_2}{\varrho_2 + \rho_2 v}. \tag{32}$$

and from Equations (25) and (28), we obtain

$$g = \frac{\alpha_1}{\pi_1} y \quad \text{and} \quad s = \frac{\alpha_2}{\pi_2} z. \tag{33}$$

Now substituting in Equations (24) and (27), we obtain

$$y = \frac{\rho_1 x v \pi_1}{\alpha_1(\pi_1 + \mu_1)} \quad \text{and} \quad z = \frac{\rho_2 w v \pi_2}{\alpha_2(\pi_2 + \mu_2)}. \tag{34}$$

Now substituting in Equation (29), we obtain

$$\left(\frac{\beta_1 \rho_1 x \pi_1}{\pi_1 + \mu_1} + \frac{\beta_2 \rho_2 w \pi_2}{\pi_2 + \mu_2} - \theta \right) v = 0. \tag{35}$$

There are two solutions for Equation (35), $v = 0$ and $\left(\frac{\beta_1 \rho_1 \pi_1}{\pi_1+\mu_1} x + \frac{\beta_2 \rho_2 \pi_2}{\pi_2+\mu_2} w - \theta \right) = 0$. When $v = 0$, we obtain $y = z = g = s = 0$, $x = \frac{\zeta_1}{\varrho_1}$ and $w = \frac{\zeta_2}{\varrho_2}$, which gives the infection-free equilibrium $EQ_0 = (x^0, 0, 0, w^0, 0, 0, 0, 0, 0)$, where

$$x^0 = \frac{\zeta_1}{\varrho_1} \quad \text{and} \quad w^0 = \frac{\zeta_2}{\varrho_2}.$$

When $v \neq 0$ and $\frac{\beta_1 \rho_1 \pi_1}{\pi_1+\mu_1} x + \frac{\beta_2 \rho_2 \pi_2}{\pi_2+\mu_2} w - \theta = 0$, then from Equation (32), we obtain

$$\frac{\beta_1 \rho_1 \pi_1 \zeta_1}{(\pi_1 + \mu_1)(\varrho_1 + \rho_1 v)} + \frac{\beta_2 \rho_2 \pi_2 \zeta_2}{(\pi_2 + \mu_2)(\varrho_2 + \rho_2 v)} - \theta = 0.$$

We define a function H as:

$$H(v) = \frac{\beta_1 \rho_1 \pi_1 \zeta_1}{(\pi_1 + \mu_1)(\varrho_1 + \rho_1 v)} + \frac{\beta_2 \rho_2 \pi_2 \zeta_2}{(\pi_2 + \mu_2)(\varrho_2 + \rho_2 v)} - \theta = 0.$$

Then

$$H(0) = \frac{\beta_1\rho_1\pi_1\zeta_1}{\varrho_1(\pi_1+\mu_1)} + \frac{\beta_2\rho_2\pi_2\zeta_2}{\varrho_2(\pi_2+\mu_2)} - \theta = \theta\left(\frac{\beta_1\rho_1\pi_1\zeta_1}{\theta\varrho_1(\pi_1+\mu_1)} + \frac{\beta_2\rho_2\pi_2\zeta_2}{\theta\varrho_2(\pi_2+\mu_2)} - 1\right) = \theta(R_0 - 1),$$

where

$$R_0 = R_{01} + R_{02},\ R_{01} = \frac{\beta_1\rho_1\pi_1\zeta_1}{\theta\varrho_1(\pi_1+\mu_1)} \text{ and } R_{02} = \frac{\beta_2\rho_2\pi_2\zeta_2}{\theta\varrho_2(\pi_2+\mu_2)}. \tag{36}$$

Thus, $H(0) > 0$, when $R_0 > 1$. The parameter R_0 represents the basic HIV-1 single-infection reproductive number.

$$\lim_{v\to\infty} H(v) = -\theta.$$

Further,

$$H'(v) = -\left(\frac{\beta_1\zeta_1\pi_1\rho_1^2}{(\pi_1+\mu_1)(\varrho_1+\rho_1 v)^2} + \frac{\beta_2\zeta_2\pi_2\rho_2^2}{(\pi_2+\mu_2)(\varrho_2+\rho_2 v)^2}\right) < 0.$$

Hence, H is a strictly decreasing function of v, and thus, there exists a unique $\hat{v} \in (0, \infty)$ such that $H(\hat{v}) = 0$. It follows that

$$\hat{x} = \frac{\zeta_1}{\varrho_1 + \rho_1\hat{v}} > 0 \text{ and } \hat{w} = \frac{\zeta_2}{\varrho_2 + \rho_2\hat{v}} > 0.$$

Then, Equations (33) and (34) give

$$\hat{y} = \frac{\pi_1\rho_1\hat{x}\,\hat{v}}{\alpha_1(\pi_1+\mu_1)} > 0,\ \hat{z} = \frac{\pi_2\rho_2\hat{w}\,\hat{v}}{\alpha_2(\pi_2+\mu_2)} > 0,\quad \hat{g} = \frac{\alpha_1}{\pi_1}\hat{y} > 0 \quad \text{and} \quad \hat{s} = \frac{\alpha_2}{\pi_2}\hat{z} > 0$$

Here, $\hat{v}$ satisfies the following quadratic equation:

$$A\hat{v}^2 + B\hat{v} + C = 0, \tag{37}$$

with

$$A = \theta\rho_1\rho_2(\pi_1+\mu_1)(\pi_2+\mu_2),$$

$$B = \theta(\pi_1+\mu_1)(\pi_2+\mu_2)(\varrho_1\rho_2 + \varrho_2\rho_1) - \rho_1\rho_2(\beta_1\zeta_1\pi_1(\pi_2+\mu_2) + \beta_2\zeta_2\pi_2(\pi_1+\mu_1)),$$

$$\begin{aligned} C &= \theta\varrho_1\varrho_2(\pi_1+\mu_1)(\pi_2+\mu_2) - \beta_1\varrho_2\rho_1\zeta_1\pi_1(\pi_2+\mu_2) - \beta_2\varrho_1\rho_2\zeta_2\pi_2(\pi_1+\mu_1) \\ &= \theta\varrho_1\varrho_2(\pi_1+\mu_1)(\pi_2+\mu_2)\left(1 - \frac{\beta_1\rho_1\zeta_1\pi_1}{\varrho_1\theta(\pi_1+\mu_1)} - \frac{\beta_2\rho_2\zeta_2\pi_2}{\varrho_2\theta(\pi_2+\mu_2)}\right) \\ &= -\theta\varrho_1\varrho_2(\pi_1+\mu_1)(\pi_2+\mu_2)(R_0 - 1). \end{aligned}$$

Obviously, $C < 0$ when $R_0 > 1$. Equation (37) has a positive root as:

$$\hat{v} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} > 0.$$

Hence, the chronic HIV-1 single-infection equilibrium $EQ_1 = (\hat{x}, \hat{g}, \hat{y}, \hat{w}, \hat{s}, \hat{z}, \hat{v}, 0, 0)$ exists when $R_0 > 1$.

Now consider $\tilde{x} = \frac{(\pi_3+\mu_3)\delta}{\rho_3\pi_3}$ and $u \neq 0$. Solving Equations (26)–(30), we obtain two equilibria: The chronic HTLV-I single-infection equilibrium $EQ_2 = (\tilde{x}, 0, 0, \tilde{w}, 0, 0, 0, \tilde{q}, \tilde{u})$, where

$$\tilde{x} = \frac{(\pi_3+\mu_3)\delta}{\rho_3\pi_3},\qquad \tilde{w} = \frac{\zeta_2}{\varrho_2} = w^0,\qquad \tilde{u} = \frac{\varrho_1}{\rho_3}(R_1 - 1),\qquad \tilde{q} = \frac{\delta\varrho_1}{\pi_3\rho_3}(R_1 - 1),$$

where

$$R_1 = \frac{\rho_3\zeta_1\pi_3}{\varrho_1\delta(\pi_3+\mu_3)}, \tag{38}$$

Parameter R_1 is the basic HTLV-I single-infection reproductive number. Consequently, EQ_2 exists when $R_1 > 1$. The other equilibrium is the chronic HIV-1/HTLV-I coinfection equilibrium $EQ_3 = (\bar{x}, \bar{g}, \bar{y}, \bar{w}, \bar{s}, \bar{z}, \bar{v}, \bar{q}, \bar{u})$, where

$$\bar{x} = \frac{(\pi_3 + \mu_3)\delta}{\rho_3 \pi_3} = \tilde{x}, \quad \bar{g} = \frac{\varrho_2 \rho_1 \delta(\pi_3 + \mu_3)}{\rho_2 \rho_3 \pi_3 (\pi_1 + \mu_1)}(R_2 - 1), \qquad \bar{y} = \frac{\pi_1 \varrho_2 \rho_1 \delta(\pi_3 + \mu_3)}{\alpha_1 \rho_2 \rho_3 \pi_3 (\pi_1 + \mu_1)}(R_2 - 1),$$
$$\bar{w} = \frac{\pi_1 \beta_1 \rho_1 \delta(\pi_2 + \mu_2)(\pi_3 + \mu_3)}{\beta_2 \rho_2 \rho_3 \pi_2 \pi_3 (\pi_1 + \mu_1)}\left(\frac{R_1}{R_{01}} - 1\right),$$
$$\bar{s} = \frac{\varrho_2 \pi_1 \beta_1 \rho_1 \delta(\pi_3 + \mu_3)}{\beta_2 \rho_2 \rho_3 \pi_2 \pi_3 (\pi_1 + \mu_1)}\left(\frac{R_1}{R_{01}} - 1\right)(R_2 - 1) = \frac{\varrho_2 \theta}{\beta_2 \rho_2 \pi_2}\left(\frac{R_{01} + R_{02} R_1}{R_1} - 1\right),$$
$$\bar{z} = \frac{\varrho_2 \beta_1 \rho_1 \pi_1 \delta(\pi_3 + \mu_3)}{\alpha_2 \beta_2 \rho_2 \rho_3 \pi_3 (\pi_1 + \mu_1)}\left(\frac{R_1}{R_{01}} - 1\right)(R_2 - 1) = \frac{\varrho_2 \theta}{\beta_2 \rho_2 \alpha_2}\left(\frac{R_{01} + R_{02} R_1}{R_1} - 1\right),$$
$$\bar{v} = \frac{\varrho_2}{\rho_2}(R_2 - 1), \quad \bar{q} = \frac{\delta \rho_1 \varrho_2}{\rho_2 \rho_3 \pi_3}[(R_2 - 1)(R_3 - 1)], \quad \bar{u} = \frac{\rho_1 \varrho_2}{\rho_2 \rho_3}[(R_2 - 1)(R_3 - 1)].$$

and

$$R_2 = \frac{\zeta_2 \beta_2 \rho_2 \rho_3 \pi_2 \pi_3 (\pi_1 + \mu_1)}{\varrho_2 \beta_1 \rho_1 \delta \pi_1 (\pi_2 + \mu_2)(\pi_3 + \mu_3)(\frac{R_1}{R_{01}} - 1)}, \ R_3 = \frac{\varrho_1 \rho_2}{\varrho_2 \rho_1}\left(\frac{R_1 - 1}{R_2 - 1}\right).$$

We can see that EQ_3 exists when $\frac{R_1}{R_{01}} > 1$, $R_2 > 1$ and $R_3 > 1$. □

5. Global Stability

In this section, we demonstrate the global asymptotic stability of all equilibria by establishing appropriate Lyapunov functions. Define a function $G(x) \geq 0$ as $G(x) = x - 1 - \ln x$. We have

$$\ln x \leq x - 1. \tag{39}$$

Theorem 1. *If $R_0 \leq 1$ and $R_1 \leq 1$, then $EQ_0 = (x^0, 0, 0, w^0, 0, 0, 0, 0, 0)$ is globally asymptotically stable (GAS) in Γ_1.*

Proof. Define a discrete Lyapunov function $\Lambda_n(x_n, g_n, y_n, w_n, s_n, z_n, v_n, q_n, u_n)$ as

$$\Lambda_n = \frac{1}{\Upsilon(h)}\left[x^0 G\left(\frac{x_n}{x^0}\right) + g_n + \frac{\pi_1 + \mu_1}{\pi_1} y_n + \frac{\beta_2 \pi_2 (\pi_1 + \mu_1)}{\beta_1 \pi_1 (\pi_2 + \mu_2)} w^0 G\left(\frac{w_n}{w^0}\right) + \frac{\beta_2 \pi_2 (\pi_1 + \mu_1)}{\beta_1 \pi_1 (\pi_2 + \mu_2)} s_n \right.$$
$$\left. + \frac{\beta_2 (\pi_1 + \mu_1)}{\beta_1 \pi_1} z_n + \frac{(\pi_1 + \mu_1)}{\beta_1 \pi_1}(1 + \Upsilon(h)\theta) v_n + q_n + \frac{(\pi_3 + \mu_3)}{\pi_3}(1 + \Upsilon(h)\delta) u_n\right].$$

Clearly, $\Lambda_n > 0$ for all $x_n > 0, g_n > 0, y_n > 0, w_n > 0, s_n > 0, z_n > 0, v_n > 0, q_n > 0, u_n > 0$. In addition, $\Lambda_n(x^0, 0, 0, w^0, 0, 0, 0, 0, 0) = 0$. Evaluating the difference $\Delta\Lambda_n = \Lambda_{n+1} - \Lambda_n$ as:

$$\Delta\Lambda_n = \Lambda_{n+1} - \Lambda_n = \frac{1}{\Upsilon(h)}\left[x^0 G\left(\frac{x_{n+1}}{x^0}\right) + g_{n+1} + \frac{\pi_1 + \mu_1}{\pi_1} y_{n+1} + \frac{\beta_2 \pi_2 (\pi_1 + \mu_1)}{\beta_1 \pi_1 (\pi_2 + \mu_2)} w^0 G\left(\frac{w_{n+1}}{w^0}\right)\right.$$
$$+ \frac{\beta_2 \pi_2 (\pi_1 + \mu_1)}{\beta_1 \pi_1 (\pi_2 + \mu_2)} s_{n+1} + \frac{\beta_2 (\pi_1 + \mu_1)}{\beta_1 \pi_1} z_{n+1} + \frac{(\pi_1 + \mu_1)}{\beta_1 \pi_1}(1 + \Upsilon(h)\theta) v_{n+1} + q_{n+1}$$
$$+ \frac{(\pi_3 + \mu_3)}{\pi_3}(1 + \Upsilon(h)\delta) u_{n+1} - x^0 G\left(\frac{x_n}{x^0}\right) - g_n - \frac{\pi_1 + \mu_1}{\pi_1} y_n - \frac{\beta_2 \pi_2 (\pi_1 + \mu_1)}{\beta_1 \pi_1 (\pi_2 + \mu_2)} w^0 G\left(\frac{w_n}{w^0}\right)$$
$$- \frac{\beta_2 \pi_2 (\pi_1 + \mu_1)}{\beta_1 \pi_1 (\pi_2 + \mu_2)} s_n - \frac{\beta_2 (\pi_1 + \mu_1)}{\beta_1 \pi_1} z_n - \frac{(\pi_1 + \mu_1)}{\beta_1 \pi_1}(1 + \Upsilon(h)\theta) v_n - q_n$$
$$\left. - \frac{(\pi_3 + \mu_3)}{\pi_3}(1 + \Upsilon(h)\delta) u_n\right]$$
$$= \frac{1}{\Upsilon(h)}\left[x^0\left(\frac{x_{n+1} - x_n}{x^0} + \ln\left(\frac{x_n}{x_{n+1}}\right)\right) + (g_{n+1} - g_n) + \frac{\pi_1 + \mu_1}{\pi_1}(y_{n+1} - y_n)\right.$$

$$
\begin{aligned}
&+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}w^0\left(\frac{w_{n+1}-w_n}{w^0}+\ln\left(\frac{w_n}{w_{n+1}}\right)\right)+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}(s_{n+1}-s_n)\\
&+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}(z_{n+1}-z_n)+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}(1+\Upsilon(h)\theta)(v_{n+1}-v_n)+(q_{n+1}-q_n)\\
&+\frac{(\pi_3+\mu_3)}{\pi_3}(1+\Upsilon(h)\delta)(u_{n+1}-u_n)\bigg].
\end{aligned}
$$

Using inequality (39), we obtain

$$
\begin{aligned}
\Delta\Lambda_n \le{}& \frac{1}{\Upsilon(h)}\left(\left(x_{n+1}-x_n+x^0\left(\frac{x_n}{x_{n+1}}-1\right)\right)+(g_{n+1}-g_n)+\frac{\pi_1+\mu_1}{\pi_1}(y_{n+1}-y_n)\right.\\
&+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(w_{n+1}-w_n+w^0\left(\frac{w_n}{w_{n+1}}-1\right)\right)+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}(s_{n+1}-s_n)\\
&+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}(z_{n+1}-z_n)+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}(1+\Upsilon(h)\theta)(v_{n+1}-v_n)+(q_{n+1}-q_n)\\
&\left.+\frac{(\pi_3+\mu_3)}{\pi_3}(1+\Upsilon(h)\delta)(u_{n+1}-u_n)\right)\\
={}& \frac{1}{\Upsilon(h)}\left(\left(1-\frac{x^0}{x_{n+1}}\right)(x_{n+1}-x_n)+(g_{n+1}-g_n)+\frac{\pi_1+\mu_1}{\pi_1}(y_{n+1}-y_n)\right.\\
&+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1-\frac{w^0}{w_{n+1}}\right)(w_{n+1}-w_n)+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}(s_{n+1}-s_n)\\
&+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}(z_{n+1}-z_n)+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}(1+\Upsilon(h)\theta)(v_{n+1}-v_n)+(q_{n+1}-q_n)\\
&\left.+\frac{(\pi_3+\mu_3)}{\pi_3}(1+\Upsilon(h)\delta)(u_{n+1}-u_n)\right).
\end{aligned}
$$

From Equations (3)–(11), we have

$$
\begin{aligned}
\Delta\Lambda_n \le{}& \left(1-\frac{x^0}{x_{n+1}}\right)(\zeta_1-\varrho_1x_{n+1}-\rho_1x_{n+1}v_n-\rho_3x_{n+1}u_n)+(\rho_1x_{n+1}v_n-(\pi_1+\mu_1)g_{n+1})\\
&+\frac{\pi_1+\mu_1}{\pi_1}(\pi_1g_{n+1}-\alpha_1y_{n+1})+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1-\frac{w^0}{w_{n+1}}\right)(\zeta_2-\varrho_2w_{n+1}-\rho_2w_{n+1}v_n)\\
&+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}(\rho_2w_{n+1}v_n-(\pi_2+\mu_2)s_{n+1})+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}(\pi_2s_{n+1}-\alpha_2z_{n+1})\\
&+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}(\beta_1\alpha_1y_{n+1}+\beta_2\alpha_2z_{n+1}-\theta v_{n+1})+\frac{\theta}{\beta_1}\frac{(\pi_1+\mu_1)}{\pi_1}(v_{n+1}-v_n)\\
&+(\rho_3x_{n+1}u_n-(\pi_3+\mu_3)q_{n+1})+\frac{\delta(\pi_3+\mu_3)}{\pi_3}(u_{n+1}-u_n)+\frac{(\pi_3+\mu_3)}{\pi_3}(\pi_3q_{n+1}-\delta u_{n+1}).
\end{aligned}
$$

Collecting terms yields

$$
\begin{aligned}
\Delta\Lambda_n \le{}& \left(1-\frac{x^0}{x_{n+1}}\right)(\zeta_1-\varrho_1x_{n+1})+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1-\frac{w^0}{w_{n+1}}\right)(\zeta_2-\varrho_2w_{n+1})\\
&+\left(\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\rho_2w^0+\rho_1x^0-\frac{\theta}{\beta_1}\frac{(\pi_1+\mu_1)}{\pi_1}\right)v_n+\left(\rho_3x^0-\frac{\delta(\pi_3+\mu_3)}{\pi_3}\right)u_n.
\end{aligned}
$$

We have $\zeta_1 = \varrho_1 x^0, \qquad \zeta_2 = \varrho_2 w^0$, then we obtain

$$\begin{aligned}\Delta\Lambda_n \leq & \left(1 - \frac{x^0}{x_{n+1}}\right)(\varrho_1 x^0 - \varrho_1 x_{n+1}) + \frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1 - \frac{w^0}{w_{n+1}}\right)(\varrho_2 w^0 - \varrho_2 w_{n+1}) \\ & + \left(\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\rho_2 w^0 + \rho_1 x^0 - \frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\right)v_n + \left(\rho_3 x^0 - \frac{\delta(\pi_3+\mu_3)}{\pi_3}\right)u_n \\ = & -\frac{\varrho_1\left(x_{n+1}-x^0\right)^2}{x_{n+1}} - \frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\frac{\varrho_2(w_{n+1}-w^0)^2}{w_{n+1}} \\ & + \frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\left(\frac{\rho_1\beta_1\zeta_1\pi_1}{\theta\varrho_1(\pi_1+\mu_1)} + \frac{\rho_2\beta_2\zeta_2\pi_2}{\theta\varrho_2(\pi_2+\mu_2)} - 1\right)v_n \\ & + \frac{\delta(\pi_3+\mu_3)}{\pi_3}\left(\frac{\rho_3\zeta_1\pi_3}{\delta\varrho_1(\pi_3+\mu_3)} - 1\right)u_n.\end{aligned}$$

From Equations (36) and (38), we can write

$$\begin{aligned}\Delta\Lambda_n \leq & -\varrho_1\frac{\left(x_{n+1}-x^0\right)^2}{x_{n+1}} - \frac{\beta_2\pi_2\varrho_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\frac{(w_{n+1}-w^0)^2}{w_{n+1}} + \frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}(R_0-1)v_n \\ & + \frac{\delta(\pi_3+\mu_3)}{\pi_3}(R_1-1)u_n.\end{aligned}$$

Since $R_0 \leq 1$ and $R_1 \leq 1$, then Λ_n is monotonically decreasing. Clearly $\Lambda_n \geq 0$, and hence, there is a limit $\lim_{n\to\infty}\Lambda_n \geq 0$ and thus $\lim_{n\to\infty}\Delta\Lambda_n = 0$, which gives $\lim_{n\to\infty} x_n = x^0$, $\lim_{n\to\infty} w_n = w^0$, $\lim_{n\to\infty}(R_0-1)v_n = 0$ and $\lim_{n\to\infty}(R_1-1)u_n = 0$. We consider four cases:

(i) $R_0 = 1$ and $R_1 = 1$, and then from Equation (6),

$$0 = \zeta_2 - \varrho_2 w^0 - \rho_2 w^0 \lim_{n\to\infty} v_n \Rightarrow \lim_{n\to\infty} v_n = 0. \tag{40}$$

In addition, from Equations (3), (4), (7) and (9), we obtain

$$0 = \zeta_1 - \varrho_1 x^0 - \rho_1 x^0 \lim_{n\to\infty} v_n - \rho_3 x^0 \lim_{n\to\infty} u_n \Rightarrow \lim_{n\to\infty} u_n = 0, \tag{41}$$

$$0 = \rho_1 x^0 \lim_{n\to\infty} v_n - (\pi_1+\mu_1)\lim_{n\to\infty} g_{n+1} \Rightarrow \lim_{n\to\infty} g_n = 0, \tag{42}$$

$$0 = \rho_2 w^0 \lim_{n\to\infty} v_n - (\pi_2+\mu_2)\lim_{n\to\infty} s_{n+1} \Rightarrow \lim_{n\to\infty} s_n = 0, \tag{43}$$

$$0 = \beta_1\alpha_1 \lim_{n\to\infty} y_{n+1} + \beta_2\alpha_2 \lim_{n\to\infty} z_{n+1} - \theta \lim_{n\to\infty} v_{n+1} \Rightarrow \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0. \tag{44}$$

Therefore, from Equation (11), we obtain

$$0 = \pi_3 \lim_{n\to\infty} q_{n+1} - \delta \lim_{n\to\infty} u_{n+1} \Rightarrow \lim_{n\to\infty} q_n = 0. \tag{45}$$

(ii) $R_0 = 1$, $R_1 < 1$ and $\lim_{n\to\infty} u_n = 0$. Equations (40) and (42)–(45) yield $\lim_{n\to\infty} v_n = 0$, $\lim_{n\to\infty} g_n = 0$, $\lim_{n\to\infty} y_n = 0$, $\lim_{n\to\infty} s_n = 0$, $\lim_{n\to\infty} z_n = 0$ and $\lim_{n\to\infty} q_n = 0$.

(iii) $R_0 < 1$, $R_1 = 1$ and $\lim_{n\to\infty} v_n = 0$. Equations (41)–(45), give $\lim_{n\to\infty} u_n = 0$, $\lim_{n\to\infty} g_n = 0$, $\lim_{n\to\infty} y_n = 0$, $\lim_{n\to\infty} s_n = 0$, $\lim_{n\to\infty} z_n = 0$ and $\lim_{n\to\infty} q_n = 0$.

(iv) $R_0 < 1$, $R_1 < 1$, $\lim_{n\to\infty} v_n = 0$ and $\lim_{n\to\infty} u_n = 0$. From Equations (42)–(45), we obtain $\lim_{n\to\infty} g_n = 0$, $\lim_{n\to\infty} y_n = 0$, $\lim_{n\to\infty} s_n = 0$, $\lim_{n\to\infty} z_n = 0$ and $\lim_{n\to\infty} q_n = 0$.

Consequently, if $R_0 \leq 1$ and $R_1 \leq 1$, then $\lim_{n\to\infty}(x_n, g_n, y_n, w_n, s_n, z_n, v_n, q_n, u_n) = (x^0, 0, 0, w^0, 0, 0, 0, 0, 0)$. This gives that EQ_0 is GAS. □

Theorem 2. *If $R_0 > 1$ and $R_1 \leq 1$ then $EQ_1 = (\hat{x}, \hat{g}, \hat{y}, \hat{w}, \hat{s}, \hat{z}, \hat{v}, 0, 0)$ is GAS in $\Gamma_1 \backslash \Gamma_0$.*

Proof. Define

$$\Theta_n = \frac{1}{\Upsilon(h)}\Bigg[\hat{x}G\left(\frac{x_n}{\hat{x}}\right) + \hat{g}G\left(\frac{g_n}{\hat{g}}\right) + \frac{(\pi_1+\mu_1)}{\pi_1}\hat{y}G\left(\frac{y_n}{\hat{y}}\right) + \frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\hat{w}G\left(\frac{w_n}{\hat{w}}\right)$$
$$+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\hat{s}G\left(\frac{s_n}{\hat{s}}\right) + \frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\hat{z}G\left(\frac{z_n}{\hat{z}}\right) + \frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\hat{v}(1+\Upsilon(h)\theta)G\left(\frac{v_n}{\hat{v}}\right) + q_n$$
$$+\frac{(\pi_3+\mu_3)}{\pi_3}(1+\Upsilon(h)\delta)u_n\Bigg].$$

Clearly $\Theta_n > 0$ for all $x_n > 0, g_n > 0, y_n > 0, w_n > 0, s_n > 0, z_n > 0, v_n > 0, q_n > 0, u_n > 0$. In addition $\Theta_n(\hat{x},\hat{g},\hat{y},\hat{w},\hat{s},\hat{z},\hat{v},0,0) = 0$. Computing the difference $\Delta\Theta_n = \Theta_{n+1} - \Theta_n$ as:

$$\Delta\Theta_n = \frac{1}{\Upsilon(h}\Bigg[\hat{x}\left(\frac{x_{n+1}}{\hat{x}} - \frac{x_n}{\hat{x}} + \ln\left(\frac{x_n}{x_{n+1}}\right)\right) + \hat{g}\left(\frac{g_{n+1}}{\hat{g}} - \frac{g_n}{\hat{g}} + \ln\left(\frac{g_n}{g_{n+1}}\right)\right)$$
$$+\frac{(\pi_1+\mu_1)}{\pi_1}\hat{y}\left(\frac{y_{n+1}}{\hat{y}} - \frac{y_n}{\hat{y}} + \ln\left(\frac{y_n}{y_{n+1}}\right)\right) + \frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\hat{w}\left(\frac{w_{n+1}}{\hat{w}} - \frac{w_n}{\hat{w}} + \ln\left(\frac{w_n}{w_{n+1}}\right)\right)$$
$$+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\hat{s}\left(\frac{s_{n+1}}{\hat{s}} - \frac{s_n}{\hat{s}} + \ln\left(\frac{s_n}{s_{n+1}}\right)\right) + \frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\hat{z}\left(\frac{z_{n+1}}{\hat{z}} - \frac{z_n}{\hat{z}} + \ln\left(\frac{z_n}{z_{n+1}}\right)\right)$$
$$+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}(1+\Upsilon(h)\theta)\hat{v}\left(\frac{v_{n+1}}{\hat{v}} - \frac{v_n}{\hat{v}} + \ln\left(\frac{v_n}{v_{n+1}}\right)\right) + (q_{n+1} - q_n)$$
$$+\frac{(\pi_3+\mu_3)}{\pi_3}(1+\Upsilon(h\delta)(u_{n+1}-u_n)\Bigg].$$

Using inequality (39), we obtain

$$\Delta\Theta_n \le \frac{1}{\Upsilon(h)}\Bigg[x_{n+1} - x_n + \hat{x}\left(\frac{x_n}{x_{n+1}} - 1\right) + g_{n+1} - g_n + \hat{g}\left(\frac{g_n}{g_{n+1}} - 1\right)$$
$$+\frac{(\pi_1+\mu_1)}{\pi_1}\left(y_{n+1} - y_n + \hat{y}\left(\frac{y_n}{y_{n+1}} - 1\right)\right) + \frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(w_{n+1} - w_n + \hat{w}\left(\frac{w_n}{w_{n+1}} - 1\right)\right)$$
$$+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(s_{n+1} - s_n + \hat{s}\left(\frac{s_n}{s_{n+1}} - 1\right)\right) + \frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\left(z_{n+1} - z_n + \hat{z}\left(\frac{z_n}{z_{n+1}} - 1\right)\right)$$
$$+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\left(v_{n+1} - v_n + \hat{v}\left(\frac{v_n}{v_{n+1}} - 1\right)\right) + \frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\Upsilon(h)\theta\left(v_{n+1} - v_n + \hat{v}\ln\left(\frac{v_n}{v_{n+1}}\right)\right)$$
$$+(q_{n+1} - q_n) + \frac{(\pi_3+\mu_3)}{\pi_3}(1+\Upsilon(h)\delta)(u_{n+1}-u_n)\Bigg]. \tag{46}$$

Inequality (46) can be written as:

$$\Delta\Theta_n \le \frac{1}{\Upsilon(h)}\Bigg[\left(1 - \frac{\hat{x}}{x_{n+1}}\right)(x_{n+1} - x_n) + \left(1 - \frac{\hat{g}}{g_{n+1}}\right)(g_{n+1} - g_n) + \frac{(\pi_1+\mu_1)}{\pi_1}\left(1 - \frac{\hat{y}}{y_{n+1}}\right)(y_{n+1} - y_n)$$
$$+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1 - \frac{\hat{w}}{w_{n+1}}\right)(w_{n+1} - w_n) + \frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1 - \frac{\hat{s}}{s_{n+1}}\right)(s_{n+1} - s_n)$$
$$+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\left(1 - \frac{\hat{z}}{z_{n+1}}\right)(z_{n+1} - z_n) + \frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\left(1 - \frac{\hat{v}}{v_{n+1}}\right)(v_{n+1} - v_n) + (q_{n+1} - q_n)$$
$$+\frac{(\pi_3+\mu_3)}{\pi_3}(1+\Upsilon(h)\delta)(u_{n+1}-u_n)\Bigg] + \frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}v_{n+1} - \frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}v_n + \frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\hat{v}\ln\left(\frac{v_n}{v_{n+1}}\right).$$

From Equations (3)–(11), we have

$$
\begin{aligned}
\Delta\Theta_n \leq{}& \left(1-\frac{\hat{x}}{x_{n+1}}\right)(\zeta_1-\varrho_1 x_{n+1}-\rho_1 x_{n+1}v_n-\rho_3 x_{n+1}u_n)+\left(1-\frac{\hat{g}}{g_{n+1}}\right)(\rho_1 x_{n+1}v_n\\
&-(\pi_1+\mu_1)g_{n+1})+\frac{(\pi_1+\mu_1)}{\pi_1}\left(1-\frac{\hat{y}}{y_{n+1}}\right)(\pi_1 g_{n+1}-\alpha_1 y_{n+1})+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1-\frac{\hat{w}}{w_{n+1}}\right)\\
&\times(\zeta_2-\varrho_2 w_{n+1}-\rho_2 w_{n+1}v_n)+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1-\frac{\hat{s}}{s_{n+1}}\right)(\rho_2 w_{n+1}v_n-(\pi_2+\mu_2)s_{n+1})\\
&+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\left(1-\frac{\hat{z}}{z_{n+1}}\right)(\pi_2 s_{n+1}-\alpha_2 z_{n+1})+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\left(1-\frac{\hat{v}}{v_{n+1}}\right)\\
&\times(\beta_1\alpha_1 y_{n+1}+\beta_2\alpha_2 z_{n+1}-\theta v_{n+1})+\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}v_{n+1}-\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}v_n+\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\hat{v}\ln\left(\frac{v_n}{v_{n+1}}\right)\\
&+\rho_3 x_{n+1}u_n-(\pi_3+\mu_3)q_{n+1}+\frac{(\pi_3+\mu_3)}{\pi_3}(\pi_3 q_{n+1}-\delta u_{n+1})+\frac{\delta(\pi_3+\mu_3)}{\pi_3}(u_{n+1}-u_n).
\end{aligned}
$$

Collecting terms, we obtain

$$
\begin{aligned}
\Delta\Theta_n \leq{}& \left(1-\frac{\hat{x}}{x_{n+1}}\right)(\zeta_1-\varrho_1 x_{n+1})+\left(\rho_1\hat{x}+\frac{\beta_2\rho_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\hat{w}-\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\right)v_n+\rho_3\hat{x}\hat{u}\frac{u_n}{\hat{u}}\\
&-\rho_1\hat{x}\hat{v}\frac{x_{n+1}v_n\hat{g}}{\hat{x}\hat{v}g_{n+1}}+(\pi_1+\mu_1)\hat{g}-(\pi_1+\mu_1)\hat{g}\frac{g_{n+1}\hat{y}}{\hat{g}y_{n+1}}+\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\hat{y}\\
&+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1-\frac{\hat{w}}{w_{n+1}}\right)(\zeta_2-\varrho_2 w_{n+1})-\frac{\beta_2\rho_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\hat{w}\hat{v}\frac{\hat{s}w_{n+1}v_n}{s_{n+1}\hat{w}\hat{v}}+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1}\hat{s}\\
&-\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1}\hat{s}\frac{\hat{z}s_{n+1}}{z_{n+1}\hat{s}}+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\hat{z}-\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\hat{y}\frac{\hat{v}y_{n+1}}{v_{n+1}\hat{y}}-\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\hat{z}\frac{\hat{v}z_{n+1}}{v_{n+1}\hat{z}}\\
&+\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\hat{v}+\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\hat{v}\ln\left(\frac{v_n}{v_{n+1}}\right)-\frac{\delta(\pi_3+\mu_3)}{\pi_3}u_n.
\end{aligned}
$$

Utilizing the following conditions for EQ_1:

$$
\begin{aligned}
&\zeta_1=\varrho_1\hat{x}+\rho_1\hat{x}\hat{v}, \qquad \rho_1\hat{x}\hat{v}=(\pi_1+\mu_1)\hat{g},\\
&\pi_1\hat{g}=\alpha_1\hat{y}, \qquad \zeta_2=\varrho_2\hat{w}+\rho_2\hat{w}\hat{v},\\
&\rho_2\hat{w}\hat{v}=(\pi_2+\mu_2)\hat{s}, \qquad \pi_2\hat{s}=\alpha_2\hat{z},\\
&\theta\hat{v}=\beta_1\alpha_1\hat{y}+\beta_2\alpha_2\hat{z},
\end{aligned}
$$

we obtain

$$
\zeta_1=\varrho_1\hat{x}+\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\hat{y}, \qquad \zeta_2=\varrho_2\hat{w}+\frac{(\pi_2+\mu_2)}{\pi_2}\alpha_2\hat{z}.
$$

Then

$$
\left(\rho_1\hat{x}+\frac{\beta_2\rho_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\hat{w}-\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\right)v_n=0,
$$

and

$$
\begin{aligned}
\Delta\Theta_n \leq{}& \left(1-\frac{\hat{x}}{x_{n+1}}\right)(\varrho_1\hat{x}-\varrho_1 x_{n+1})+\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\hat{y}-\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\hat{y}\frac{\hat{x}}{x_{n+1}}+\rho_3\hat{x}\hat{u}\frac{u_n}{\hat{u}}\\
&-\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\hat{y}\frac{x_{n+1}v_n\hat{g}}{\hat{x}\hat{v}g_{n+1}}+\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\hat{y}-\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\hat{y}\frac{g_{n+1}\hat{y}}{\hat{g}y_{n+1}}+\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\hat{y}\\
&+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1-\frac{\hat{w}}{w_{n+1}}\right)(\varrho_2\hat{w}-\varrho_2 w_{n+1})+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\hat{z}-\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\hat{z}\frac{\hat{w}}{w_{n+1}}\\
&-\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\hat{z}\frac{\hat{s}w_{n+1}v_n}{s_{n+1}\hat{w}\hat{v}}+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\hat{z}-\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\hat{z}\frac{\hat{z}s_{n+1}}{z_{n+1}\hat{s}}+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\hat{z}
\end{aligned}
$$

$$-\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\hat{y}\frac{\hat{v}y_{n+1}}{v_{n+1}\hat{y}}-\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\hat{z}\frac{\hat{v}z_{n+1}}{v_{n+1}\hat{z}}+\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\hat{y}+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\hat{z}$$
$$+\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\hat{y}\ln\left(\frac{v_n}{v_{n+1}}\right)+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\hat{z}\ln\left(\frac{v_n}{v_{n+1}}\right)-\frac{\delta(\pi_3+\mu_3)}{\pi_3}u_n. \tag{47}$$

Inequality (47) takes the form

$$\Delta\Theta_n\le-\varrho_1\frac{(x_{n+1}-\hat{x})^2}{x_{n+1}}-\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\frac{\varrho_2(w_{n+1}-\hat{w})^2}{w_{n+1}}+\left(\rho_3\hat{x}-\frac{\delta(\pi_3+\mu_3)}{\pi_3}\right)u_n$$
$$+\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\hat{y}\left[4-\frac{\hat{x}}{x_{n+1}}-\frac{x_{n+1}v_n\hat{g}}{\hat{x}\hat{v}g_{n+1}}-\frac{g_{n+1}\hat{y}}{\hat{g}y_{n+1}}-\frac{\hat{v}y_{n+1}}{v_{n+1}\hat{y}}+\ln\left(\frac{v_n}{v_{n+1}}\right)\right]$$
$$+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\hat{z}\left[4-\frac{\hat{w}}{w_{n+1}}-\frac{\hat{s}w_{n+1}v_n}{s_{n+1}\hat{w}\hat{v}}-\frac{\hat{z}s_{n+1}}{z_{n+1}\hat{s}}-\frac{\hat{v}z_{n+1}}{v_{n+1}\hat{z}}+\ln\left(\frac{v_n}{v_{n+1}}\right)\right].$$

Using the following equalities:

$$\ln\left(\frac{v_n}{v_{n+1}}\right)=\ln\left(\frac{\hat{x}}{x_{n+1}}\right)+\ln\left(\frac{x_{n+1}v_n\hat{g}}{\hat{x}\hat{v}g_{n+1}}\right)+\ln\left(\frac{g_{n+1}\hat{y}}{\hat{g}y_{n+1}}\right)+\ln\left(\frac{\hat{v}y_{n+1}}{v_{n+1}\hat{y}}\right), \tag{48}$$
$$\ln\left(\frac{v_n}{v_{n+1}}\right)=\ln\left(\frac{\hat{w}}{w_{n+1}}\right)+\ln\left(\frac{\hat{s}w_{n+1}v_n}{s_{n+1}\hat{w}\hat{v}}\right)+\ln\left(\frac{\hat{z}s_{n+1}}{z_{n+1}\hat{s}}\right)+\ln\left(\frac{\hat{v}z_{n+1}}{v_{n+1}\hat{z}}\right). \tag{49}$$

We obtain

$$\Delta\Theta_n\le-\varrho_1\frac{(x_{n+1}-\hat{x})^2}{x_{n+1}}-\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\frac{\varrho_2(w_{n+1}-\hat{w})^2}{w_{n+1}}+\left(\frac{\rho_3\zeta_1}{\varrho_1+\rho_1\hat{v}}-\frac{\delta(\pi_3+\mu_3)}{\pi_3}\right)u_n$$
$$-\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\hat{y}\left[G\left(\frac{\hat{x}}{x_{n+1}}\right)+G\left(\frac{x_{n+1}v_n\hat{g}}{\hat{x}\hat{v}g_{n+1}}\right)+G\left(\frac{g_{n+1}\hat{y}}{\hat{g}y_{n+1}}\right)+G\left(\frac{\hat{v}y_{n+1}}{v_{n+1}\hat{y}}\right)\right]$$
$$-\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\hat{z}\left[G\left(\frac{\hat{w}}{w_{n+1}}\right)+G\left(\frac{\hat{s}w_{n+1}v_n}{s_{n+1}\hat{w}\hat{v}}\right)+G\left(\frac{\hat{z}s_{n+1}}{z_{n+1}\hat{s}}\right)+G\left(\frac{\hat{v}z_{n+1}}{v_{n+1}\hat{z}}\right)\right].$$

Since $R_0>1$, then $\hat{v}>0$ and $\frac{\rho_3\zeta_1}{\varrho_1+\rho_1\hat{v}}<\frac{\rho_3\zeta_1}{\varrho_1}$. Therefore, we obtain

$$\Delta\Theta_n\le-\varrho_1\frac{(x_{n+1}-\hat{x})^2}{x_{n+1}}-\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\frac{\varrho_2(w_{n+1}-\hat{w})^2}{w_{n+1}}+\frac{\delta(\pi_3+\mu_3)}{\pi_3}(R_1-1)u_n$$
$$-\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\hat{y}\left[G\left(\frac{\hat{x}}{x_{n+1}}\right)+G\left(\frac{x_{n+1}v_n\hat{g}}{\hat{x}\hat{v}g_{n+1}}\right)+G\left(\frac{g_{n+1}\hat{y}}{\hat{g}y_{n+1}}\right)+G\left(\frac{\hat{v}y_{n+1}}{v_{n+1}\hat{y}}\right)\right]$$
$$-\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\hat{z}\left[G\left(\frac{\hat{w}}{w_{n+1}}\right)+G\left(\frac{\hat{s}w_{n+1}v_n}{s_{n+1}\hat{w}\hat{v}}\right)+G\left(\frac{\hat{z}s_{n+1}}{z_{n+1}\hat{s}}\right)+G\left(\frac{\hat{v}z_{n+1}}{v_{n+1}\hat{z}}\right)\right].$$

Since $R_0>1$ and if $R_1\le 1$, then Θ_n is monotonically decreasing. We have $\Theta_n\ge 0$, and then there is a limit $\lim_{n\to\infty}\Theta_n\ge 0$ and hence $\lim_{n\to\infty}\Delta\Theta_n=0$, which implies $\lim_{n\to\infty}x_n=\hat{x}$, $\lim_{n\to\infty}g_n=\hat{g}$, $\lim_{n\to\infty}y_n=\hat{y}$, $\lim_{n\to\infty}w_n=\hat{w}$, $\lim_{n\to\infty}s_n=\hat{s}$, $\lim_{n\to\infty}z_n=\hat{z}$, $\lim_{n\to\infty}v_n=\hat{v}$, and $\lim_{n\to\infty}(R_1-1)u_n=0$. Now, we address two cases:

(i) $R_1=1$. From Equation (3), we obtain

$$0=\zeta_1-\varrho_1\hat{x}-\rho_1\hat{x}\,\hat{v}-\rho_3\hat{x}\lim_{n\to\infty}u_n\Rightarrow\lim_{n\to\infty}u_n=0.$$

From Equation (11), we obtain

$$0=\pi_3\lim_{n\to\infty}q_{n+1}-\delta\lim_{n\to\infty}u_{n+1}\Rightarrow\lim_{n\to\infty}q_n=0. \tag{50}$$

(ii) $R_1<1$ and then $\lim_{n\to\infty}u_n=0$. From Equation (50), we obtain $\lim_{n\to\infty}q_n=0$. Hence, EQ_1 is GAS. □

Theorem 3. *If $R_1 > 1$ and $R_{02} + \frac{R_{01}}{R_1} \leq 1$, then $EQ_2 = (\tilde{x}, 0, 0, \tilde{w}, 0, 0, 0, \tilde{q}, \tilde{u})$ is GAS in $\Gamma_1 \backslash \Gamma_0$.*

Proof. Consider a discrete Lyapunov function Φ_n as:

$$\begin{aligned}\Phi_n = \frac{1}{\Upsilon(h)}\Bigg[&\tilde{x}G\left(\frac{x_n}{\tilde{x}}\right) + g_n + \frac{(\pi_1+\mu_1)}{\pi_1}y_n + \frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\tilde{w}G\left(\frac{w_n}{\tilde{w}}\right)\\ &+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}s_n + \frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}z_n + \frac{(\pi_1+\mu_1)}{\beta_1\pi_1}(1+\Upsilon(h)\theta)v_n\\ &+\tilde{q}G\left(\frac{q_n}{\tilde{q}}\right) + \frac{(\pi_3+\mu_3)}{\pi_3}(1+\Upsilon(h)\delta)\tilde{u}G\left(\frac{u_n}{\tilde{u}}\right)\Bigg].\end{aligned}$$

Computing the difference $\Delta\Phi_n = \Phi_{n+1} - \Phi_n$ as:

$$\begin{aligned}\Delta\Phi_n = \frac{1}{\Upsilon(h)}\Bigg[&\tilde{x}\left(\frac{x_{n+1}}{\tilde{x}} - \frac{x_n}{\tilde{x}} + \ln\left(\frac{x_n}{x_{n+1}}\right)\right) + (g_{n+1}-g_n) + \frac{(\pi_1+\mu_1)}{\pi_1}(y_{n+1}-y_n)\\ &+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\tilde{w}\left(\frac{w_{n+1}}{\tilde{w}} - \frac{w_n}{\tilde{w}} + \ln\left(\frac{w_n}{w_{n+1}}\right)\right) + \frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}(s_{n+1}-s_n)\\ &+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}(z_{n+1}-z_n) + \frac{(\pi_1+\mu_1)}{\beta_1\pi_1}(v_{n+1}-v_n) + \tilde{q}\left(\frac{q_{n+1}}{\tilde{q}} - \frac{q_n}{\tilde{q}} + \ln\left(\frac{q_n}{q_{n+1}}\right)\right)\\ &+\frac{(\pi_3+\mu_3)}{\pi_3}\tilde{u}\left(\frac{u_{n+1}}{\tilde{u}} - \frac{u_n}{\tilde{u}} + \ln\left(\frac{u_n}{u_{n+1}}\right)\right)\Bigg] + \frac{(\pi_3+\mu_3)}{\pi_3}\delta\tilde{u}\left(G\left(\frac{u_{n+1}}{\tilde{u}}\right) - G\left(\frac{u_n}{\tilde{u}}\right)\right)\\ &+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\theta v_{n+1} - \frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\theta v_n.\end{aligned}$$

Using inequality (39), we obtain

$$\begin{aligned}\Delta\Phi_n \leq \frac{1}{\Upsilon(h)}\Bigg[&\tilde{x}\left(\frac{x_{n+1}-x_n}{\tilde{x}} + \frac{x_n}{x_{n+1}} - 1\right) + (g_{n+1}-g_n) + \frac{(\pi_1+\mu_1)}{\pi_1}(y_{n+1}-y_n)\\ &+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\tilde{w}\left(\frac{w_{n+1}-w_n}{\tilde{w}} + \frac{w_n}{w_{n+1}} - 1\right) + \frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}(s_{n+1}-s_n)\\ &+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}(z_{n+1}-z_n) + \frac{(\pi_1+\mu_1)}{\beta_1\pi_1}(v_{n+1}-v_n) + \tilde{q}\left(\frac{q_{n+1}-q_n}{\tilde{q}} + \frac{q_n}{q_{n+1}} - 1\right)\\ &+\frac{(\pi_3+\mu_3)}{\pi_3}\tilde{u}\left(\frac{u_{n+1}-u_n}{\tilde{u}} + \frac{u_n}{u_{n+1}} - 1\right)\Bigg] + \frac{(\pi_3+\mu_3)}{\pi_3}\delta\tilde{u}\left(G\left(\frac{u_{n+1}}{\tilde{u}}\right) - G\left(\frac{u_n}{\tilde{u}}\right)\right)\\ &+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\theta v_{n+1} - \frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\theta v_n.\end{aligned} \tag{51}$$

We write inequality (51) as:

$$\begin{aligned}\Delta\Phi_n \leq \frac{1}{\Upsilon(h)}\Bigg[&\left(1 - \frac{\tilde{x}}{x_{n+1}}\right)(x_{n+1}-x_n) + (g_{n+1}-g_n) + \frac{(\pi_1+\mu_1)}{\pi_1}(y_{n+1}-y_n)\\ &+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1 - \frac{\tilde{w}}{w_{n+1}}\right)(w_{n+1}-w_n) + \frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}(s_{n+1}-s_n)\\ &+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}(z_{n+1}-z_n) + \frac{(\pi_1+\mu_1)}{\beta_1\pi_1}(v_{n+1}-v_n) + \left(1 - \frac{\tilde{q}}{q_{n+1}}\right)(q_{n+1}-q_n)\\ &+\frac{(\pi_3+\mu_3)}{\pi_3}\left(1 - \frac{\tilde{u}}{u_{n+1}}\right)(u_{n+1}-u_n)\Bigg] + \frac{(\pi_3+\mu_3)}{\pi_3}\delta\tilde{u}\left(G\left(\frac{u_{n+1}}{\tilde{u}}\right) - G\left(\frac{u_n}{\tilde{u}}\right)\right)\\ &+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\theta v_{n+1} - \frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\theta v_n\end{aligned}$$

From Equations (3)–(11), we have

$$\Delta\Phi_n \le \left(1-\frac{\tilde{x}}{x_{n+1}}\right)(\zeta_1-\varrho_1 x_{n+1}-\rho_1 x_{n+1}v_n-\rho_3 x_{n+1}u_n)+(\rho_1 x_{n+1}v_n-(\pi_1+\mu_1)g_{n+1})$$
$$+\frac{(\pi_1+\mu_1)}{\pi_1}(\pi_1 g_{n+1}-\alpha_1 y_{n+1})+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1-\frac{\tilde{w}}{w_{n+1}}\right)(\zeta_2-\varrho_2 w_{n+1}-\rho_2 w_{n+1}v_n$$
$$+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}(\rho_2 w_{n+1}v_n-(\pi_2+\mu_2)s_{n+1})+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}(\pi_2 s_{n+1}-\alpha_2 z_{n+1})$$
$$+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}(\beta_1\alpha_1 y_{n+1}+\beta_2\alpha_2 z_{n+1}-\theta v_{n+1})+\left(1-\frac{\tilde{q}}{q_{n+1}}\right)(\rho_3 x_{n+1}u_n$$
$$-(\pi_3+\mu_3)q_{n+1})+\frac{(\pi_3+\mu_3)}{\pi_3}\left(1-\frac{\tilde{u}}{u_{n+1}}\right)(\pi_3 q_{n+1}-\delta u_{n+1})$$
$$+\frac{(\pi_3+\mu_3)}{\pi_3}\delta\tilde{u}\left(\frac{u_{n+1}}{\tilde{u}}-\frac{u_n}{\tilde{u}}+\ln\left(\frac{u_n}{u_{n+1}}\right)\right)+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\theta v_{n+1}-\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\theta v_n.$$

By collecting the terms, we obtain

$$\Delta\Phi_n \le \left(1-\frac{\tilde{x}}{x_{n+1}}\right)(\zeta_1-\varrho_1 x_{n+1})+\rho_3\tilde{x}u_n+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1-\frac{\tilde{w}}{w_{n+1}}\right)(\zeta_2-\varrho_2 w_{n+1})$$
$$-\rho_3 x_{n+1}u_n\frac{\tilde{q}}{q_{n+1}}+(\pi_3+\mu_3)\tilde{q}-(\pi_3+\mu_3)q_{n+1}\frac{\tilde{u}}{u_{n+1}}+\frac{(\pi_3+\mu_3)}{\pi_3}\delta\tilde{u}-\frac{(\pi_3+\mu_3)}{\pi_3}\delta u_n$$
$$+\left(\rho_1\tilde{x}+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\rho_2\tilde{w}-\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\theta\right)v_n+\frac{(\pi_3+\mu_3)}{\pi_3}\delta\tilde{u}\ln\left(\frac{u_n}{u_{n+1}}\right).$$

Using the equilibria conditions of EQ_2

$$\zeta_1=\varrho_1\tilde{x}+\rho_3\tilde{x}\tilde{u},\quad \zeta_2=\varrho_2\tilde{w},\qquad \rho_3\tilde{x}\tilde{u}=\frac{(\pi_3+\mu_3)}{\pi_3}\delta\tilde{u}.$$

We obtain

$$\Delta\Phi_n \le -\frac{\varrho_1}{x_{n+1}}(x_{n+1}-\tilde{x})^2-\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\frac{\varrho_2}{w_{n+1}}(w_{n+1}-\tilde{w})^2$$
$$+\left(\rho_1\tilde{x}+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\rho_2\tilde{w}-\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\right)v_n$$
$$+\frac{(\pi_3+\mu_3)}{\pi_3}\delta\tilde{u}\left(3-\frac{\tilde{x}}{x_{n+1}}-\frac{\tilde{q}x_{n+1}u_n}{\tilde{x}\tilde{u}q_{n+1}}-\frac{\tilde{u}q_{n+1}}{\tilde{q}u_{n+1}}+\ln\left(\frac{u_n}{u_{n+1}}\right)\right). \tag{52}$$

Since we have

$$\rho_1\tilde{x}+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\rho_2\tilde{w}-\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}$$
$$=\frac{\rho_1(\pi_3+\mu_3)\delta}{\rho_3\pi_3}+\frac{\beta_2\pi_2(\pi_1+\mu_1)\rho_2\zeta_2}{\beta_1\pi_1\varrho_2(\pi_2+\mu_2)}-\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}$$
$$=\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\left(\frac{\rho_1\beta_1\pi_1\delta(\pi_3+\mu_3)}{\rho_3\pi_3\theta(\pi_1+\mu_1)}+\frac{\beta_2\pi_2\rho_2\zeta_2}{\varrho_2\theta(\pi_2+\mu_2)}-1\right)$$
$$=\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\left(\frac{R_{01}}{R_1}+R_{02}-1\right),$$

and using the following equality:

$$\ln\left(\frac{u_n}{u_{n+1}}\right)=\ln\left(\frac{\tilde{x}}{x_{n+1}}\right)+\ln\left(\frac{\tilde{q}x_{n+1}u_n}{\tilde{x}\tilde{u}q_{n+1}}\right)+\ln\left(\frac{q_{n+1}\tilde{u}}{\tilde{q}u_{n+1}}\right). \tag{53}$$

Then Equation (52) becomes

$$\Delta\Phi_n \leq -\varrho_1\frac{(x_{n+1}-\tilde{x})^2}{x_{n+1}} - \frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\frac{\varrho_2(w_{n+1}-\tilde{w})^2}{w_{n+1}} + \frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\left(R_{02}+\frac{R_{01}}{R_1}-1\right)v_n$$
$$-\frac{(\pi_3+\mu_3)}{\pi_3}\delta\tilde{u}\left(G\left(\frac{\tilde{x}}{x_{n+1}}\right)+G\left(\frac{\tilde{q}x_{n+1}u_n}{\tilde{x}\tilde{u}q_{n+1}}\right)+G\left(\frac{q_{n+1}\tilde{u}}{\tilde{q}u_{n+1}}\right)\right).$$

Since, $R_{02}+\frac{R_{01}}{R_1}\leq 1$, then $\Delta\Phi_n \leq 0$, for all $n \geq 0$. Hence, the sequence Φ_n is monotonically decreasing. Since $\Phi_n \geq 0$, then $\lim_{n\to\infty}\Phi_n \geq 0$ and thus $\lim_{n\to\infty}\Delta\Phi_n = 0$. Thus, $\lim_{n\to\infty} x_n = \tilde{x}$, $\lim_{n\to\infty} w_n = \tilde{w}$, $\lim_{n\to\infty} q_n = \tilde{q}$, $\lim_{n\to\infty} u_n = \tilde{u}$ and $\lim_{n\to\infty}\left(R_{02}+\frac{R_{01}}{R_1}-1\right)v_n = 0$. We have two cases:

(i) $R_{02}+\frac{R_{01}}{R_1}=1$, and from Equation (3)

$$0=\zeta_1-\varrho_1\tilde{x}-\rho_1\tilde{x}\lim_{n\to\infty}v_n-\rho_3\tilde{x}\,\tilde{u} \implies \lim_{n\to\infty}v_n=0. \tag{54}$$

Moreover, from Equations (9), (5) and (8),

$$0=\beta_1\alpha_1\lim_{n\to\infty}y_{n+1}+\beta_2\alpha_2\lim_{n\to\infty}z_{n+1}=0 \implies \lim_{n\to\infty}y_n=0 \text{ and } \lim_{n\to\infty}z_n=0, \tag{55}$$
$$0=\pi_1\lim_{n\to\infty}g_{n+1} \implies \lim_{n\to\infty}g_n=0, \tag{56}$$
$$0=\pi_2\lim_{n\to\infty}s_{n+1} \implies \lim_{n\to\infty}s_n=0. \tag{57}$$

(ii) $R_{02}+\frac{R_{01}}{R_1}<1$ and $\lim_{n\to\infty}v_n=0$. Equations (55)–(57) imply that $\lim_{n\to\infty}y_n=0$, $\lim_{n\to\infty}z_n=0$, $\lim_{n\to\infty}g_n=0$ and $\lim_{n\to\infty}s_n=0$. This proves that EQ_2 is GAS. □

Theorem 4. *If* $\frac{R_1}{R_{01}}>1$, $R_2>1$ *and* $R_3>1$, *then* $EQ_3=(\bar{x},\bar{g},\bar{y},\bar{w},\bar{s},\bar{z},\bar{v},\bar{q},\bar{u})$ *is GAS in the interior of* Γ_1.

Proof. Consider

$$\Psi_n=\frac{1}{\Upsilon(h)}\left[\bar{x}G\left(\frac{x_n}{\bar{x}}\right)+\bar{g}G\left(\frac{g_n}{\bar{g}}\right)+\frac{(\pi_1+\mu_1)}{\pi_1}\bar{y}G\left(\frac{y_n}{\bar{y}}\right)+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\bar{w}G\left(\frac{w_n}{\bar{w}}\right)\right.$$
$$+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\bar{s}G\left(\frac{s_n}{\bar{s}}\right)+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{z}G\left(\frac{z_n}{\bar{z}}\right)+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}(1+\Upsilon(h)\theta)\bar{v}G\left(\frac{v_n}{\bar{v}}\right)$$
$$\left.+\bar{q}G\left(\frac{q_n}{\bar{q}}\right)+\frac{(\pi_3+\mu_3)}{\pi_3}(1+\Upsilon(h)\delta)\bar{u}G\left(\frac{u_n}{\bar{u}}\right)\right].$$

Computing the difference $\Delta\Psi_n=\Psi_{n+1}-\Psi_n$ as:

$$\Delta\Psi_n=\frac{1}{\Upsilon(h)}\left[\bar{x}\left(\frac{x_{n+1}}{\bar{x}}-\frac{x_n}{\bar{x}}+\ln\left(\frac{x_n}{x_{n+1}}\right)\right)+\bar{g}\left(\frac{g_{n+1}}{\bar{g}}-\frac{g_n}{\bar{g}}+\ln\left(\frac{g_n}{g_{n+1}}\right)\right)\right.$$
$$+\frac{(\pi_1+\mu_1)}{\pi_1}\bar{y}\left(\frac{y_{n+1}}{\bar{y}}-\frac{y_n}{\bar{y}}+\ln\left(\frac{y_n}{y_{n+1}}\right)\right)+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\bar{w}\left(\frac{w_{n+1}}{\bar{w}}-\frac{w_n}{\bar{w}}+\ln\left(\frac{w_n}{w_{n+1}}\right)\right)$$
$$+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\bar{s}\left(\frac{s_{n+1}}{\bar{s}}-\frac{s_n}{\bar{s}}+\ln\left(\frac{s_n}{s_{n+1}}\right)\right)+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{z}\left(\frac{z_{n+1}}{\bar{z}}-\frac{z_n}{\bar{z}}+\ln\left(\frac{z_n}{z_{n+1}}\right)\right)$$
$$+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{v}\left(\frac{v_{n+1}}{\bar{v}}-\frac{v_n}{\bar{v}}+\ln\left(\frac{v_n}{v_{n+1}}\right)\right)+\bar{q}\left(\frac{q_{n+1}}{\bar{q}}-\frac{q_n}{\bar{q}}+\ln\left(\frac{q_n}{q_{n+1}}\right)\right)$$
$$\left.+\frac{(\pi_3+\mu_3)}{\pi_3}\bar{u}\left(\frac{u_{n+1}}{\bar{u}}-\frac{u_n}{\bar{u}}+\ln\left(\frac{u_n}{u_{n+1}}\right)\right)\right]+\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{v}\left(G\left(\frac{v_{n+1}}{\bar{v}}\right)-G\left(\frac{v_n}{\bar{v}}\right)\right)$$
$$+\frac{\delta(\pi_3+\mu_3)}{\pi_3}\bar{u}\left(G\left(\frac{u_{n+1}}{\bar{u}}\right)-G\left(\frac{u_n}{\bar{u}}\right)\right).$$

Using inequality (39), we obtain

$$\begin{aligned}\Delta\Psi_n \leq \frac{1}{\Upsilon(h)}\Bigg[&\bar{x}\left(\frac{x_{n+1}-x_n}{\bar{x}}+\frac{x_n}{x_{n+1}}-1\right)+\bar{g}\left(\frac{g_{n+1}-g_n}{\bar{g}}+\frac{g_n}{g_{n+1}}-1\right)\\
&+\frac{(\pi_1+\mu_1)}{\pi_1}\bar{y}\left(\frac{y_{n+1}-y_n}{\bar{y}}+\frac{y_n}{y_{n+1}}-1\right)+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\bar{w}\left(\frac{w_{n+1}-w_n}{\bar{w}}+\frac{w_n}{w_{n+1}}-1\right)\\
&+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\bar{s}\left(\frac{s_{n+1}-s_n}{\bar{s}}+\frac{s_n}{s_{n+1}}-1\right)+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{z}\left(\frac{z_{n+1}-z_n}{\bar{z}}+\frac{z_n}{z_{n+1}}-1\right)\\
&+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{v}\left(\frac{v_{n+1}-v_n}{\bar{v}}+\frac{v_n}{v_{n+1}}-1\right)+\bar{q}\left(\frac{q_{n+1}-q_n}{\bar{q}}+\frac{q_n}{q_{n+1}}-1\right)\\
&+\frac{(\pi_3+\mu_3)}{\pi_3}\bar{u}\left(\frac{u_{n+1}-u_n}{\bar{u}}+\frac{u_n}{u_{n+1}}-1\right)\Bigg]+\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{v}\left(G\left(\frac{v_{n+1}}{\bar{v}}\right)-G\left(\frac{v_n}{\bar{v}}\right)\right)\\
&+\frac{\delta(\pi_3+\mu_3)}{\pi_3}\bar{u}\left(G\left(\frac{u_{n+1}}{\bar{u}}\right)-G\left(\frac{u_n}{\bar{u}}\right)\right).\end{aligned} \tag{58}$$

We write inequality (58) as:

$$\begin{aligned}\Delta\Psi_n \leq \frac{1}{\Upsilon(h)}\Bigg[&\left(1-\frac{\bar{x}}{x_{n+1}}\right)(x_{n+1}-x_n)+\left(1-\frac{\bar{g}}{g_{n+1}}\right)(g_{n+1}-g_n)\\
&+\frac{(\pi_1+\mu_1)}{\pi_1}\left(1-\frac{\bar{y}}{y_{n+1}}\right)(y_{n+1}-y_n)+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1-\frac{\bar{w}}{w_{n+1}}\right)(w_{n+1}-w_n)\\
&+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1-\frac{\bar{s}}{s_{n+1}}\right)(s_{n+1}-s_n)+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\left(1-\frac{\bar{z}}{z_{n+1}}\right)(z_{n+1}-z_n)\\
&+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\left(1-\frac{\bar{v}}{v_{n+1}}\right)(v_{n+1}-v_n)+\left(1-\frac{\bar{q}}{q_{n+1}}\right)(q_{n+1}-q_n)\\
&+\frac{(\pi_3+\mu_3)}{\pi_3}\left(1-\frac{\bar{u}}{u_{n+1}}\right)(u_{n+1}-u_n)\Bigg]+\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{v}\left(\frac{v_{n+1}}{\bar{v}}-\frac{v_n}{\bar{v}}+\ln\left(\frac{v_n}{v_{n+1}}\right)\right)\\
&+\frac{\delta(\pi_3+\mu_3)}{\pi_3}\bar{u}\left(\frac{u_{n+1}}{\bar{u}}-\frac{u_n}{\bar{u}}+\ln\left(\frac{u_n}{u_{n+1}}\right)\right).\end{aligned}$$

From Equations (3)–(11), we have

$$\begin{aligned}\Delta\Psi_n \leq &\left(1-\frac{\bar{x}}{x_{n+1}}\right)(\zeta_1-\varrho_1 x_{n+1}-\rho_1 x_{n+1}v_n-\rho_3 x_{n+1}u_n)+\left(1-\frac{\bar{g}}{g_{n+1}}\right)\\
&\times(\rho_1 x_{n+1}v_n-(\pi_1+\mu_1)g_{n+1})+\frac{(\pi_1+\mu_1)}{\pi_1}\left(1-\frac{\bar{y}}{y_{n+1}}\right)(\pi_1 g_{n+1}-\alpha_1 y_{n+1})\\
&+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1-\frac{\bar{w}}{w_{n+1}}\right)(\zeta_2-\varrho_2 w_{n+1}-\rho_2 w_{n+1}v_n)\\
&+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1-\frac{\bar{s}}{s_{n+1}}\right)(\rho_2 w_{n+1}v_n-(\pi_2+\mu_2)s_{n+1})\\
&+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\left(1-\frac{\bar{z}}{z_{n+1}}\right)(\pi_2 s_{n+1}-\alpha_2 z_{n+1})\\
&+\frac{(\pi_1+\mu_1)}{\beta_1\pi_1}\left(1-\frac{\bar{v}}{v_{n+1}}\right)(\beta_1\alpha_1 y_{n+1}+\beta_2\alpha_2 z_{n+1}-\theta v_{n+1})\\
&+\left(1-\frac{\bar{q}}{q_{n+1}}\right)(\rho_3 x_{n+1}u_n-(\pi_3+\mu_3)q_{n+1})+\frac{(\pi_3+\mu_3)}{\pi_3}\left(1-\frac{\bar{u}}{u_{n+1}}\right)(\pi_3 q_{n+1}-\delta u_{n+1})\\
&+\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}v_{n+1}-\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}v_n+\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{v}\ln\left(\frac{v_n}{v_{n+1}}\right)+\frac{\delta(\pi_3+\mu_3)}{\pi_3}u_{n+1}\\
&-\frac{\delta(\pi_3+\mu_3)}{\pi_3}u_n+\frac{\delta(\pi_3+\mu_3)}{\pi_3}\bar{u}\ln\left(\frac{u_n}{u_{n+1}}\right).\end{aligned}$$

After collecting the terms, we obtain

$$\begin{aligned}
\Delta\Psi_n \leq & \left(1-\frac{\bar{x}}{x_{n+1}}\right)(\zeta_1-\varrho_1 x_{n+1})+\rho_1\bar{x}v_n+\rho_3\bar{x}u_n-\rho_1 x_{n+1}v_n\frac{\bar{g}}{g_{n+1}}+(\pi_1+\mu_1)\bar{g} \\
& -(\pi_1+\mu_1)g_{n+1}\frac{\bar{y}}{y_{n+1}}+\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\bar{y}+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1-\frac{\bar{w}}{w_{n+1}}\right)(\zeta_2-\varrho_2 w_{n+1}) \\
& +\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\rho_2\bar{w}v_n-\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\rho_2 w_{n+1}v_n\frac{\bar{s}}{s_{n+1}}+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{s} \\
& -\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1}s_{n+1}\frac{\bar{z}}{z_{n+1}}+\frac{\beta_2\alpha_2(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{z}-\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1 y_{n+1}\frac{\bar{v}}{v_{n+1}} \\
& -\frac{\beta_2\alpha_2(\pi_1+\mu_1)}{\beta_1\pi_1}\frac{z_{n+1}\bar{v}}{v_{n+1}}+\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{v}-\rho_3 x_{n+1}u_n\frac{\bar{q}}{q_{n+1}} \\
& +(\pi_3+\mu_3)\bar{q}-(\pi_3+\mu_3)\frac{q_{n+1}\bar{u}}{u_{n+1}}+\frac{\delta(\pi_3+\mu_3)}{\pi_3}\bar{u}-\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}v_n \\
& +\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{v}\ln\left(\frac{v_n}{v_{n+1}}\right)-\frac{\delta(\pi_3+\mu_3)}{\pi_3}u_n+\frac{\delta(\pi_3+\mu_3)}{\pi_3}\bar{u}\ln\left(\frac{u_n}{u_{n+1}}\right).
\end{aligned}$$

Using the equilibrium conditions for EQ_3

$$\begin{aligned}
\rho_1\bar{x}\bar{v} &= (\pi_1+\mu_1)\bar{g}, & \rho_3\bar{x}\bar{u} &= (\pi_3+\mu_3)\bar{q}, \\
\rho_2\bar{w}\bar{v} &= (\pi_2+\mu_2)\bar{s}, & \theta\bar{v} &= \beta_1\alpha_1\bar{y}+\beta_2\alpha_2\bar{z}, \\
\zeta_1 &= \varrho_1\bar{x}+\rho_1\bar{x}\bar{v}+\rho_3\bar{x}\bar{u}, & \zeta_2 &= \varrho_2\bar{w}+\rho_2\bar{w}\bar{v}, \\
\bar{q} &= \frac{\delta\bar{u}}{\pi_3}, \qquad \bar{g}=\frac{\alpha_1\bar{y}}{\pi_1}, & \bar{s} &= \frac{\alpha_2\bar{z}}{\pi_2},
\end{aligned}$$

we obtain

$$\zeta_1=\varrho_1\bar{x}+\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\bar{y}+\frac{(\pi_3+\mu_3)}{\pi_3}\delta\bar{u}, \qquad \zeta_2=\varrho_2\bar{w}+\frac{(\pi_2+\mu_2)}{\pi_2}\alpha_2\bar{z}.$$

Then

$$\begin{aligned}
\Delta\Psi_n \leq & \left(1-\frac{\bar{x}}{x_{n+1}}\right)\left(\varrho_1\bar{x}+\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\bar{y}+\frac{(\pi_3+\mu_3)}{\pi_3}\delta\bar{u}-\varrho_1 x_{n+1}\right)+\rho_1\bar{x}v_n \\
& +\rho_3\bar{x}u_n-\rho_1\bar{x}\bar{v}\frac{x_{n+1}v_n\bar{g}}{\bar{x}\bar{v}g_{n+1}}+(\pi_1+\mu_1)\bar{g}-\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\bar{y}\frac{g_{n+1}\bar{y}}{\bar{g}y_{n+1}}+\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\bar{y} \\
& +\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\left(1-\frac{\bar{w}}{w_{n+1}}\right)\left(\varrho_2\bar{w}+\frac{(\pi_2+\mu_2)}{\pi_2}\alpha_2\bar{z}-\varrho_2 w_{n+1}\right)+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\rho_2\bar{w}v_n \\
& -\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\rho_2\bar{w}\bar{v}\frac{w_{n+1}v_n\bar{s}}{\bar{w}\bar{v}s_{n+1}}+\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{s}-\frac{\beta_2\pi_2(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{s}\frac{\bar{z}s_{n+1}}{\bar{s}z_{n+1}} \\
& +\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\bar{z}-\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\bar{y}\frac{\bar{v}y_{n+1}}{\bar{y}v_{n+1}}-\frac{\beta_2\alpha_2(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{z}\frac{z_{n+1}\bar{v}}{\bar{z}v_{n+1}} \\
& +\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\bar{y}+\frac{\beta_2\alpha_2(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{z}-\rho_3\bar{x}\bar{u}\frac{x_{n+1}u_n\bar{q}}{\bar{x}\bar{u}q_{n+1}}+(\pi_3+\mu_3)\bar{q} \\
& -(\pi_3+\mu_3)\bar{q}\frac{q_{n+1}\bar{u}}{\bar{q}u_{n+1}}+\frac{(\pi_3+\mu_3)}{\pi_3}\delta\bar{u}-\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}v_n+\frac{\theta(\pi_1+\mu_1)}{\beta_1\pi_1}\bar{v}\ln\left(\frac{v_n}{v_{n+1}}\right) \\
& -\frac{\delta(\pi_3+\mu_3)}{\pi_3}u_n+\frac{\delta(\pi_3+\mu_3)}{\pi_3}\bar{u}\ln\left(\frac{u_n}{u_{n+1}}\right),
\end{aligned}$$

and we obtain

$$\begin{aligned}\Delta\Psi_n \leq& -\varrho_1\frac{(x_{n+1}-\bar{x})^2}{x_{n+1}} - \frac{\beta_2\pi_2\varrho_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\frac{(w_{n+1}-\bar{w})^2}{w_{n+1}}\\ &+\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\bar{y}\left[4-\frac{\bar{x}}{x_{n+1}}-\frac{x_{n+1}v_n\bar{g}}{\bar{x}\bar{v}g_{n+1}}-\frac{g_{n+1}\bar{y}}{\bar{g}y_{n+1}}+\frac{y_{n+1}\bar{v}}{\bar{y}v_{n+1}}+\ln\left(\frac{v_n}{v_{n+1}}\right)\right]\\ &+\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\bar{z}\left[4-\frac{\bar{w}}{w_{n+1}}-\frac{w_{n+1}v_n\bar{s}}{\bar{w}\bar{v}s_{n+1}}-\frac{\bar{z}s_{n+1}}{\bar{s}z_{n+1}}+\frac{z_{n+1}\bar{v}}{\bar{z}v_{n+1}}+\ln\left(\frac{v_n}{v_{n+1}}\right)\right]\\ &+\frac{\delta(\pi_3+\mu_3)}{\pi_3}\bar{u}\left[3-\frac{\bar{x}}{x_{n+1}}-\frac{x_{n+1}u_n\bar{q}}{\bar{x}\bar{u}q_{n+1}}-\frac{q_{n+1}\bar{u}}{\bar{q}u_{n+1}}+\ln\left(\frac{u_n}{u_{n+1}}\right)\right].\end{aligned}$$

Using equalities similar to Equations (48), (49) and (53), we obtain

$$\begin{aligned}\Delta\Psi_n \leq& -\varrho_1\frac{(x_{n+1}-\bar{x})^2}{x_{n+1}} - \frac{\beta_2\pi_2\varrho_2(\pi_1+\mu_1)}{\beta_1\pi_1(\pi_2+\mu_2)}\frac{(w_{n+1}-\bar{w})^2}{w_{n+1}}\\ &-\frac{(\pi_1+\mu_1)}{\pi_1}\alpha_1\bar{y}\left[G\left(\frac{\bar{x}}{x_{n+1}}\right)+G\left(\frac{x_{n+1}v_n\bar{g}}{\bar{x}\bar{v}g_{n+1}}\right)+G\left(\frac{g_{n+1}\bar{y}}{\bar{g}y_{n+1}}\right)+G\left(\frac{y_{n+1}\bar{v}}{\bar{y}v_{n+1}}\right)\right]\\ &-\frac{\beta_2(\pi_1+\mu_1)}{\beta_1\pi_1}\alpha_2\bar{z}\left[G\left(\frac{\bar{w}}{w_{n+1}}\right)+G\left(\frac{w_{n+1}v_n\bar{s}}{\bar{w}\bar{v}s_{n+1}}\right)+G\left(\frac{\bar{z}s_{n+1}}{\bar{s}z_{n+1}}\right)+G\left(\frac{z_{n+1}\bar{v}}{\bar{z}v_{n+1}}\right)\right]\\ &-\frac{\delta(\pi_3+\mu_3)}{\pi_3}\bar{u}\left[G\left(\frac{\bar{x}}{x_{n+1}}\right)+G\left(\frac{x_{n+1}u_n\bar{q}}{\bar{x}\bar{u}q_{n+1}}\right)+G\left(\frac{q_{n+1}\bar{u}}{\bar{q}u_{n+1}}\right)\right].\end{aligned}$$

We note that $\Delta\Psi_n \leq 0$. Hence, the sequence Ψ_n is monotonically decreasing. Since $\Psi_n \geq 0$, then $\lim_{n\to\infty}\Psi_n \geq 0$ and thus, $\lim_{n\to\infty}\Delta\Psi_n = 0$. Thus, $\lim_{n\to\infty}(x_n, g_n, y_n, w_n, s_n, z_n, v_n, q_n, u_n) = (\bar{x}, \bar{g}, \bar{y}, \bar{w}, \bar{s}, \bar{z}, \bar{v}, \bar{q}, \bar{u})$. Hence, EQ_3 is GAS. □

6. Numerical Simulations

In this section, we execute numerical simulations for the discrete-time model (3)–(11). Moreover, we discussed the impact of latent reservoirs on the HIV-1 and HTLV-I co-dynamics. We use the values of the parameters given in Table 1. Some of these values are taken from previous works for HIV-1 and HTLV-I single-infections. The values of other parameters are assumed just to execute the numerical simulations. The reason for that is the unavailability of real data from HIV-1 and HTLV-I coinfection patients. However, when the real data are available, then the values of the parameters of the coinfection model can be estimated.

Table 1. Model parameters.

Parameter	Value	Source
ζ_1	10 cells mm^{-3} day^{-1}	[27,35,54]
ζ_2	0.03198 cells mm^{-3} day^{-1}	[28,29]
α_1	0.5 day^{-1}	[20,22]
α_2	0.1 day^{-1}	[13,55]
γ_1	0.01 day^{-1}	[27,56]
γ_2	0.01 day^{-1}	[28,29]
β_1	6 viruses cells^{-1}	[55]
β_2	6 viruses cells^{-1}	[55]
θ	2 day^{-1}	[13,57]
δ	0.2 day^{-1}	[13,34]
h	0.1	[58]

Table 1. *Cont.*

Parameter	Value	Source
ρ_1	(varied) viruses^{-1} mm^3 day^{-1}	Assumed
ρ_2	(varied) viruses^{-1} mm^3 day^{-1}	Assumed
ρ_3	(varied) cells^{-1} mm^3 day^{-1}	Assumed
μ_1	0.02 day^{-1}	[54,59]
μ_2	0.01 day^{-1}	[55]
μ_3	0.01 day^{-1}	[35]
π_1	0.2 day^{-1}	[59]
π_2	0.01 day^{-1}	Assumed
π_3	0.03 day^{-1}	[60]

6.1. Stability of the Equilibria

To support the global stability results given Theorems 1–4, we show that the solutions of the discrete-time system starting from any point (any disease stage) in the feasible region will tend to one of the four equilibria. Let us consider three different initial values as:

$$\begin{aligned} \text{IV1}: &\quad (x_0, g_0, y_0, w_0, s_0, z_0, v_0, q_0, u_0) = (850, 25, 5.5, 2, 1.5, 0.1, 20, 55, 35), \\ \text{IV2}: &\quad (x_0, g_0, y_0, w_0, s_0, z_0, v_0, q_0, u_0) = (650, 20, 3.5, 1.5, 1, 0.15, 15, 45, 25), \\ \text{IV3}: &\quad (x_0, g_0, y_0, w_0, s_0, z_0, v_0, q_0, u_0) = (350, 15, 2, 1, 0.5, 0.2, 0.4, 35, 15). \end{aligned}$$

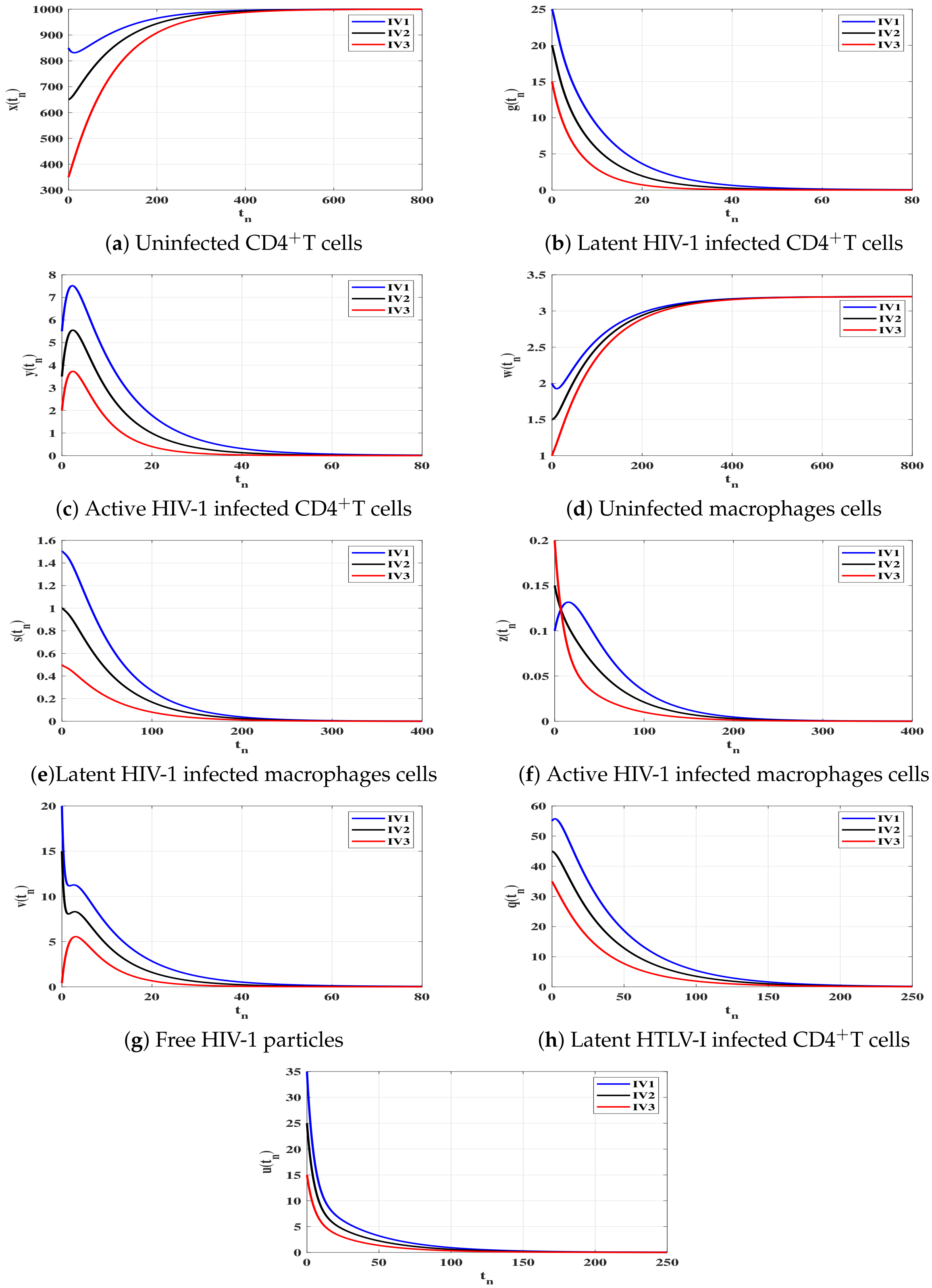
We choose ρ_1, ρ_2 and ρ_3 as:

Case (I) $\rho_1 = 0.0002$, $\rho_2 = 0.001$ and $\rho_3 = 0.0001$. This gives $R_0 = 0.550252 \leq 1$ and $R_1 = 0.375 \leq 1$. Figure 1 illustrates that the concentrations of uninfected CD4$^+$T cells and uninfected macrophages increase and tend to the healthy values $x^0 = 1000$ and $w^0 = 3.1980$, while the concentrations of other populations decrease and converge to zero. Therefore, EQ_0 is GAS, and this agrees with the result of Theorem 1. In this case, both HIV-1 and HTLV-I are cleared from the human body, regardless of the starting states .

Case (II) $\rho_1 = 0.0007$, $\rho_2 = 0.001$ and $\rho_3 = 0.0001$. These values give $R_0 = 1.91389 > 1$ and $R_1 = 0.375 \leq 1$. From Figure 2, we see that the solutions of the discrete-time model tend to the equilibrium $EQ_1 = (522.71, 21.695, 8.678, 1.388, 0.905, 0.091, 13.044, 0, 0)$. As a result, EQ_1 exists, and based on Theorem 2, it is GAS. This result shows that the HIV-1 single-infection can be reached for all initial states.

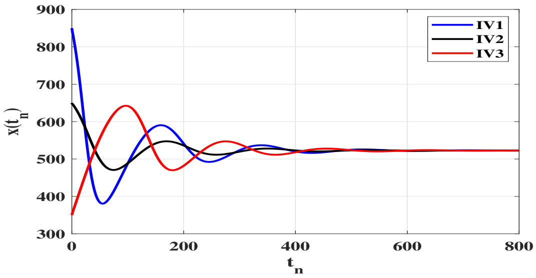
Case (III) $\rho_1 = 0.0003$, $\rho_2 = 0.0001$ and $\rho_3 = 0.00045$, and then $R_1 = 1.6875 > 1$ and $R_{02} + (R_{01}/R_1) = 0.489645 \leq 1$. Figure 3 demonstrates that the solutions of the discrete-time model reach the equilibrium $EQ_2 = (592.593, 0, 0, 3.198, 0, 0, 0, 101.852, 15.278)$ for all the initial states. According to Lemma 2 and Theorem 3, EQ_2 exists and it is GAS. This result shows that the HTLV-I single-infection can be reached for all starting states.

Case (IV) $\rho_1 = 0.00054$, $\rho_2 = 0.03$ and $\rho_3 = 0.0004$, and thus, $R_1/R_{01} = 1.01852 > 1$, $R_2 = 7.91505 > 1$, and $R_3 = 4.017 > 1$. Figure 4 illustrates that the solutions of the discrete-time model starting with initial values IV1-IV3 converge to the equilibrium $EQ_3 = (666.667, 3.772, 1.509, 0.404, 1.397, 0.1396, 2.305, 62.588, 9.388)$. Based on Lemma 2 and Theorem 4, EQ_3 exists and it is GAS. This result shows that the HIV-1 and HTLV-I coinfection can be reached for all starting states.

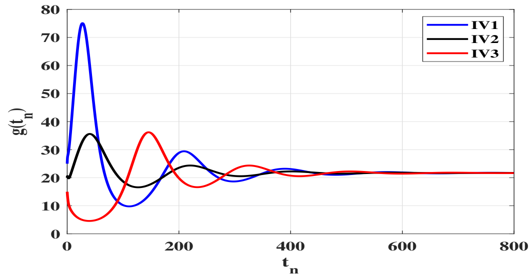


(**a**) Uninfected CD4$^+$T cells
(**b**) Latent HIV-1 infected CD4$^+$T cells
(**c**) Active HIV-1 infected CD4$^+$T cells
(**d**) Uninfected macrophages cells
(**e**)Latent HIV-1 infected macrophages cells
(**f**) Active HIV-1 infected macrophages cells
(**g**) Free HIV-1 particles
(**h**) Latent HTLV-I infected CD4$^+$T cells
(**i**) Active HTLV-I- infected CD4$^+$T cells

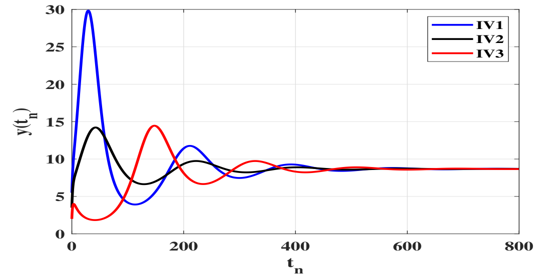
Figure 1. Solutions of systems (3)–(11) with initial conditions IV1–IV3 in the case of $R_0 \leq 1$ and $R_1 \leq 1$.



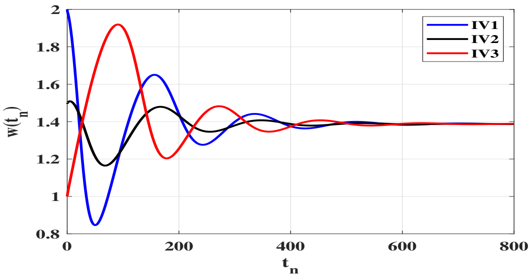
(**a**) Uninfected CD4$^+$T cells



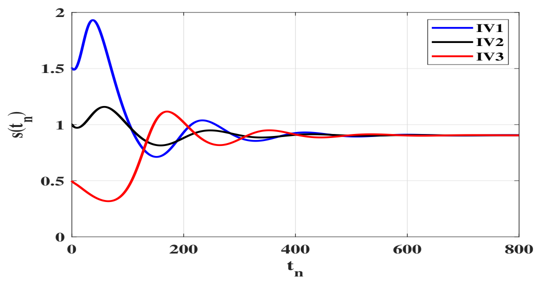
(**b**) Latent HIV-1 infected CD4$^+$T cells



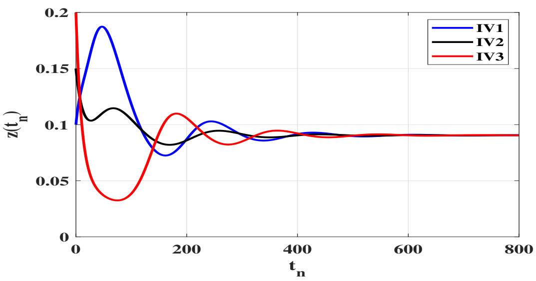
(**c**) Active HIV-1 infected CD4$^+$T cells



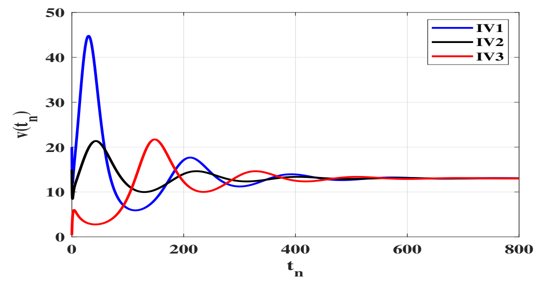
(**d**) Uninfected macrophages cells



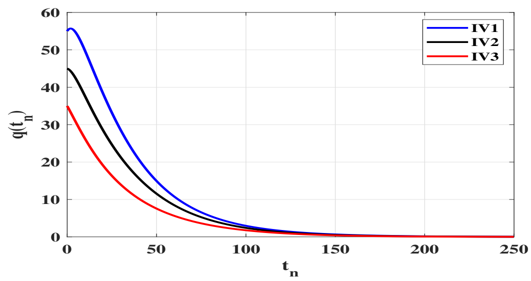
(**e**) Latent HIV-1 infected macrophages cells



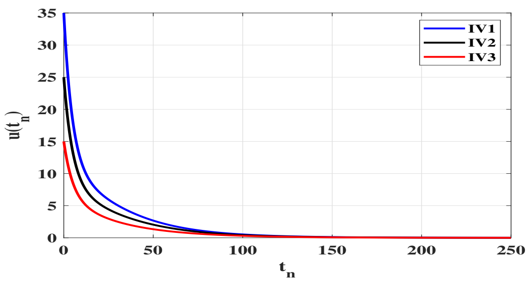
(**f**) Active HIV-1 infected macrophages cells



(**g**) Free HIV-1 particles

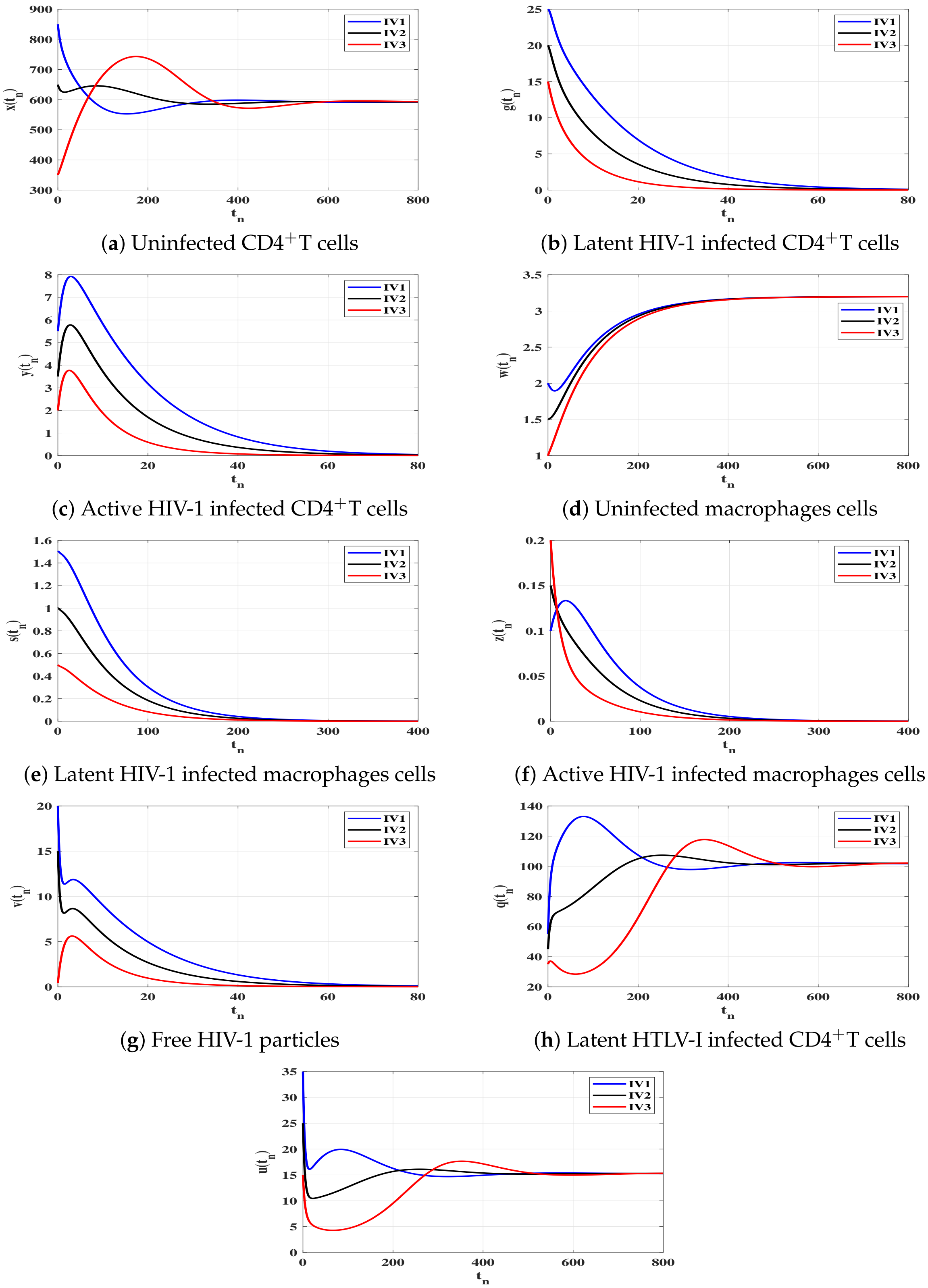


(**h**) Latent HTLV-I infected CD4$^+$T cells



(**i**) Active HTLV-I- infected CD4$^+$T cells

Figure 2. Solutions of systems (3)–(11) with initial conditions IV1–IV3 in the case of $R_0 > 1$ and $R_1 \leq 1$.



(**a**) Uninfected CD4$^+$T cells (**b**) Latent HIV-1 infected CD4$^+$T cells

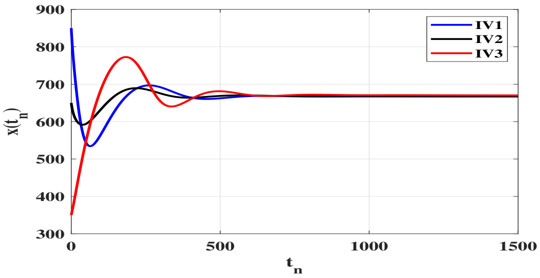
(**c**) Active HIV-1 infected CD4$^+$T cells (**d**) Uninfected macrophages cells

(**e**) Latent HIV-1 infected macrophages cells (**f**) Active HIV-1 infected macrophages cells

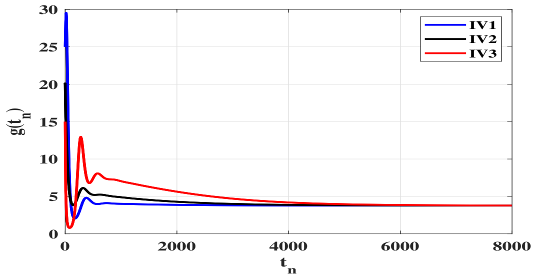
(**g**) Free HIV-1 particles (**h**) Latent HTLV-I infected CD4$^+$T cells

(**i**) Active HTLV-I- infected CD4$^+$T cells

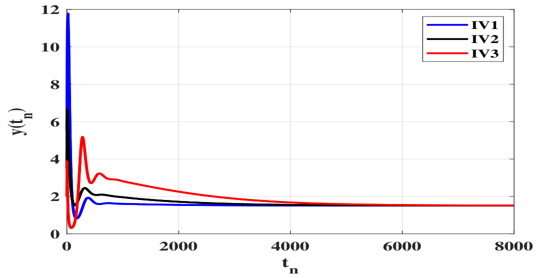
Figure 3. Solutions of systems (3)–(11) with initial conditions IV1–IV3 in the case of $R_1 > 1$ and $R_{02} + \frac{R_{01}}{R_1} \leq 1$.



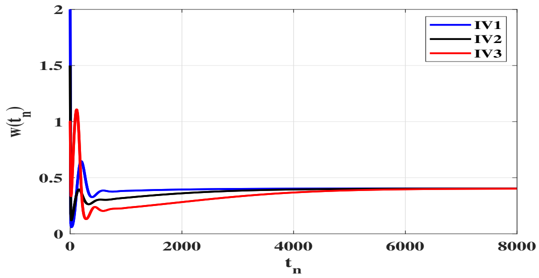
(**a**) Uninfected CD4$^+$T cells



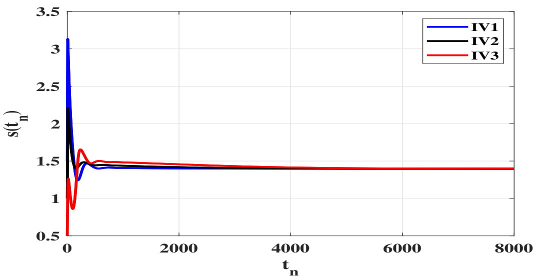
(**b**) Latent HIV-1 infected CD4$^+$T cells



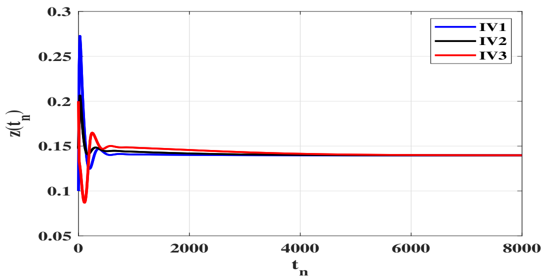
(**c**) Active HIV-1 infected CD4$^+$T cells



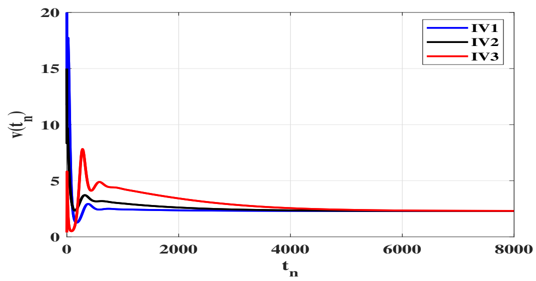
(**d**) Uninfected macrophages cells



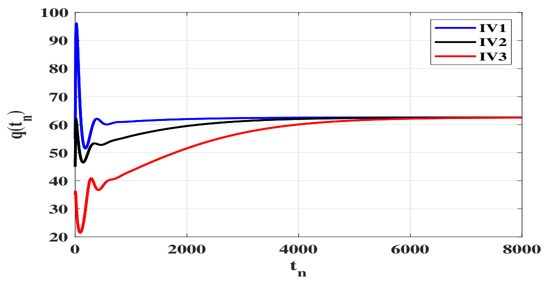
(**e**) Latent HIV-1 infected macrophages cells



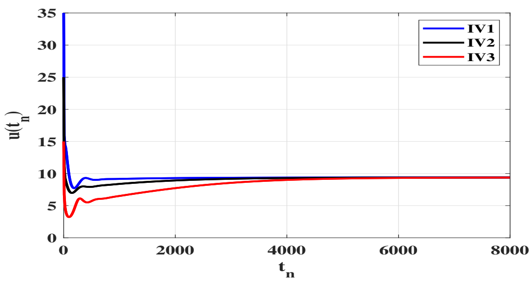
(**f**) Active HIV-1 infected macrophages cells



(**g**) Free HIV-1 particles



(**h**) Latent HTLV-I infected CD4$^+$T cells



(**i**) Active HTLV-I- infected CD4$^+$T cells

Figure 4. Solutions of systems (3)–(11) with initial conditions IV1–IV3 in the case of $\frac{R_1}{R_{01}} > 1$, $R_2 > 1$ and $R_3 > 1$.

For more confirmation, we examine the local stability of the equilibria of the discrete-time model in Cases (I)–(IV). The Jacobian matrix $J = J(x, g, y, w, s, z, v, q, u)$ of model (9)–(25) is calculated as:

$$J = \begin{pmatrix} J_{11} & 0 & 0 & 0 & 0 & 0 & J_{17} & 0 & J_{19} \\ J_{21} & J_{22} & 0 & 0 & 0 & 0 & J_{27} & 0 & J_{29} \\ J_{31} & J_{32} & J_{33} & 0 & 0 & 0 & J_{37} & 0 & J_{39} \\ 0 & 0 & 0 & J_{44} & 0 & 0 & J_{47} & 0 & 0 \\ 0 & 0 & 0 & J_{54} & J_{55} & 0 & J_{57} & 0 & 0 \\ 0 & 0 & 0 & J_{64} & J_{65} & J_{66} & J_{67} & 0 & 0 \\ J_{71} & J_{72} & J_{73} & J_{74} & J_{75} & J_{76} & J_{77} & 0 & J_{79} \\ J_{81} & 0 & 0 & 0 & 0 & 0 & J_{87} & J_{88} & J_{89} \\ J_{91} & 0 & 0 & 0 & 0 & 0 & J_{97} & J_{98} & J_{99} \end{pmatrix},$$

where

$$J_{11} = \frac{1}{1 + \Upsilon(h)(\varrho_1 + \rho_1 v + \rho_3 u)},$$

$$J_{17} = -\frac{\rho_1 \Upsilon(h)(x + \zeta_1 \Upsilon(h))}{(1 + \Upsilon(h)(\varrho_1 + \rho_1 v + \rho_3 u))^2},$$

$$J_{19} = -\frac{\Upsilon(h)(x + \zeta_1 \Upsilon(h))\rho_3}{(1 + \Upsilon(h)(\varrho_1 + \rho_1 v + \rho_3 u))^2},$$

$$J_{21} = \frac{v\Upsilon(h)\rho_1}{(1 + \Upsilon(h)(\pi_1 + \mu_1))(1 + \Upsilon(h)(\varrho_1 + \rho_1 v + \rho_3 u))},$$

$$J_{22} = \frac{1}{1 + \Upsilon(h)(\pi_1 + \mu_1)},$$

$$J_{27} = \frac{\rho_1 \Upsilon(h)(x + \zeta_1 \Upsilon(h))(1 + \Upsilon(h)\varrho_1 + \Upsilon(h)u\rho_3)}{(1 + \Upsilon(h)(\pi_1 + \mu_1))(1 + \Upsilon(h)(\varrho_1 + \rho_1 v + \rho_3 u))^2},$$

$$J_{29} = -\frac{v\Upsilon^2(h)(x + \zeta_1 \Upsilon(h))\rho_1\rho_3}{(1 + \Upsilon(h)(\pi_1 + \mu_1))(1 + \Upsilon(h)(\varrho_1 + \rho_1 v + \rho_3 u))^2},$$

$$J_{31} = \frac{v\pi_1 \Upsilon^2(h)\rho_1}{(1 + \alpha_1 \Upsilon(h))(1 + \Upsilon(h)(\pi_1 + \mu_1))(1 + \Upsilon(h)(\varrho_1 + \rho_1 v + \rho_3 u))},$$

$$J_{32} = \frac{\pi_1 \Upsilon(h)}{(1 + \alpha_1 \Upsilon(h))(1 + \Upsilon(h)(\pi_1 + \mu_1))},$$

$$J_{33} = \frac{1}{1 + \alpha_1 \Upsilon(h)},$$

$$J_{37} = \frac{\pi_1 \Upsilon^2(h)(x + \zeta_1 \Upsilon(h))\rho_1(1 + \Upsilon(h)\varrho_1 + \Upsilon(h)\rho_3 u)}{(1 + \alpha_1 \Upsilon(h))(1 + \Upsilon(h)(\pi_1 + \mu_1))(1 + \Upsilon(h)(\varrho_1 + \rho_1 v + \rho_3 u))^2},$$

$$J_{39} = -\frac{v\pi_1 \Upsilon^3(h)(x + \zeta_1 \Upsilon(h))\rho_1\rho_3}{(1 + \alpha_1 \Upsilon(h))(1 + \Upsilon(h)(\pi_1 + \mu_1))(1 + \Upsilon(h)(\varrho_1 + \rho_1 v + \rho_3 u))^2},$$

$$J_{44} = \frac{1}{1 + \Upsilon(h)\varrho_2 + \Upsilon(h)\rho_2 v},$$

$$J_{47} = -\frac{\Upsilon(h)(w + \Upsilon(h)\zeta_2)\rho_2}{(1 + \Upsilon(h)\varrho_2 + \Upsilon(h)\rho_2 v)^2},$$

$$J_{54} = \frac{\Upsilon(h)\rho_2 v}{(1 + \Upsilon(h)(\pi_2 + \mu_2))(1 + \Upsilon(h)\varrho_2 + \Upsilon(h)\rho_2 v)},$$

$$J_{55} = \frac{1}{1 + \Upsilon(h)(\pi_2 + \mu_2)},$$

$$J_{57} = \frac{\Upsilon(h)(1 + \Upsilon(h)\varrho_2)(w + \zeta_2 \Upsilon(h))\rho_2}{(1 + \Upsilon(h)(\pi_2 + \mu_2))(1 + \Upsilon(h)\varrho_2 + \Upsilon(h)\rho_2 v)^2},$$

$$J_{64} = \frac{\Upsilon^2(h)v\pi_2\rho_2}{(1+\Upsilon(h)\alpha_2)(1+\Upsilon(h)(\pi_2+\mu_2))(1+\Upsilon(h)\varrho_2+\Upsilon(h)\rho_2 v)},$$
$$J_{65} = \frac{\Upsilon(h)\pi_2}{(1+\Upsilon(h)\alpha_2)(1+\Upsilon(h)(\pi_2+\mu_2))},$$
$$J_{66} = \frac{1}{(1+\Upsilon(h)\alpha_2)},$$
$$J_{67} = \frac{\Upsilon^2(h)\pi_2(w+\Upsilon(h)\zeta_2)(\rho_2+\Upsilon(h)\varrho_2\rho_2)}{(1+\Upsilon(h)\alpha_2)(1+\Upsilon(h)(\pi_2+\mu_2))(1+\Upsilon(h)\varrho_2+\Upsilon(h)\rho_2 v)^2},$$
$$J_{71} = \frac{\Upsilon^3(h)v\alpha_1\beta_1\pi_1\rho_1}{(1+\Upsilon(h)\alpha_1)(1+\Upsilon(h)\theta)(1+\Upsilon(h)(\pi_1+\mu_1))(1+\Upsilon(h)(\varrho_1+\rho_1 v+\rho_3 u))},$$
$$J_{72} = \frac{\Upsilon^2(h)\alpha_1\beta_1\pi_1}{(1+\Upsilon(h)\alpha_1)(1+\Upsilon(h)\theta)(1+\Upsilon(h)(\pi_1+\mu_1))},$$
$$J_{73} = \frac{\Upsilon(h)\alpha_1\beta_1}{(1+\Upsilon(h)\alpha_1)(1+\Upsilon(h)\theta)},$$
$$J_{74} = \frac{\Upsilon^3(h)v\alpha_2\beta_2\pi_2\rho_2}{(1+\Upsilon(h)\alpha_2)(1+\Upsilon(h)\theta)(1+\Upsilon(h)(\pi_2+\mu_2))(1+\Upsilon(h)\varrho_2+\Upsilon(h)\rho_2 v)},$$
$$J_{75} = \frac{\Upsilon^2(h)\alpha_2\beta_2\pi_2}{(1+\Upsilon(h)\alpha_2)(1+\Upsilon(h)\theta)(1+\Upsilon(h)(\pi_2+\mu_2))},$$
$$J_{76} = \frac{\Upsilon(h)\alpha_2\beta_2}{(1+\Upsilon(h)\alpha_2)(1+\Upsilon(h)\theta)},$$
$$J_{77} = \frac{1}{1+\Upsilon(h)\theta} + \frac{\Upsilon^3(h)}{1+\Upsilon(h)\theta}\left[\frac{\alpha_2\beta_2\pi_2\rho_2(w+\Upsilon(h)\zeta_2)(1+\Upsilon(h)\varrho_2)}{(1+\Upsilon(h)\alpha_2)(1+\Upsilon(h)(\pi_2+\mu_2))(1+\Upsilon(h)\varrho_2+\Upsilon(h)\rho_2 v)^2}\right.$$
$$\left.+\frac{\alpha_1\beta_1\pi_1(x+\Upsilon(h)\zeta_1)\rho_1(1+\Upsilon(h)\varrho_1+\Upsilon(h)\rho_3 u)}{(1+\Upsilon(h)\alpha_1)(1+\Upsilon(h)(\pi_1+\mu_1))(1+\Upsilon(h)(\varrho_1+v\rho_1+\Upsilon(h)\rho_2 u))^2}\right],$$
$$J_{79} = -\frac{\Upsilon^4(h)v\alpha_1\beta_1\pi_1(x+\Upsilon(h)\zeta_1)\rho_1\rho_3}{(1+\Upsilon(h)\alpha_1)(1+\Upsilon(h)\theta)(1+\Upsilon(h)(\pi_1+\mu_1))(1+\Upsilon(h)(\varrho_1+\rho_1 v+\rho_3 u))^2},$$
$$J_{81} = \frac{\Upsilon(h)u\rho_3}{(1+\Upsilon(h)(\pi_3+\mu_3))(1+\Upsilon(h)(\varrho_1+\rho_1 v+\rho_3 u))},$$
$$J_{87} = -\frac{\Upsilon^2(h)(x+\Upsilon(h)\zeta_1)\rho_1\rho_3}{(1+\Upsilon(h)(\pi_3+\mu_3))(1+\Upsilon(h)(\varrho_1+\rho_1 v+\rho_3 u))^2},$$
$$J_{88} = \frac{1}{1+\Upsilon(h)(\pi_3+\mu_3)},$$
$$J_{89} = \frac{\rho_3\Upsilon(h)(x+\Upsilon(h)\zeta_1)(1+\Upsilon(h)\varrho_1+\Upsilon(h)\rho_1 v)}{(1+\Upsilon(h)(\pi_3+\mu_3))(1+\Upsilon(h)(\varrho_1+\rho_1 v+\rho_3 u))^2},$$
$$J_{91} = \frac{\Upsilon^2(h)u\pi_3\rho_3}{(1+\Upsilon(h)\delta)(1+\Upsilon(h)(\pi_3+\mu_3))(1+\Upsilon(h)(\varrho_1+\rho_1 v+\rho_3 u))},$$
$$J_{97} = -\frac{\Upsilon^3(h)u\pi_3(x+\zeta_1\Upsilon(h))\rho_1\rho_3}{(1+\Upsilon(h)\delta)(1+\Upsilon(h)(\pi_3+\mu_3))(1+\Upsilon(h)(\varrho_1+\rho_1 v+\rho_3 u))^2},$$
$$J_{98} = \frac{\Upsilon(h)\pi_3}{(1+\Upsilon(h)\delta)(1+\Upsilon(h)(\pi_3+\mu_3))},$$
$$J_{99} = \frac{1}{1+\Upsilon(h)\delta}\left[1+\frac{\Upsilon^2(h)\pi_3(x+\zeta_1\Upsilon(h))(1+\Upsilon(h)(\varrho_1+\rho_1 v))\rho_3}{(1+\Upsilon(h)(\pi_2+\mu_2))(1+\Upsilon(h)(\varrho_1+\rho_1 v+\rho_3 u))^2}\right],$$

Then, we compute the eigenvalues $\lambda_j, j = 1, 2, \ldots, 9$ of the matrix J, at each equilibrium. An equilibrium point of the discrete-time model is locally asymptotically stable (LAS) when $|\lambda_j| < 1$, for all $j = 1, 2, \ldots, 9$. We compute the eigenvalues of all nonnegative equilibria using the values of ρ_1, ρ_2 and ρ_3 given in Cases (I)-(IV). Table 2 contains the nonnegative equilibria, the absolute value of the eigenvalues and whether the equilibrium point is LAS or unstable. We note that when an equilibrium point is GAS, then it is also LAS, and all the other equilibria will be unstable.

Table 2. Local stability of equilibria.

Case	Equilibrium Point	$\lvert\lambda_j\rvert, j = 1,2,\ldots,9$	Stability
(I)	$EQ_0 = (1000, 0, 0, 3.20, 0, 0, 0, 0, 0)$	$(0.999, 0.999, 0.998, 0.998, 0.993, 0.990, 0.979, 0.934, 0.837)$	LAS
(II)	$EQ_0 = (1000, 0, 0, 3.20, 0, 0, 0, 0, 0)$	$(1.011, 0.999, 0.999, 0.998, 0.998, 0.990, 0.979, 0.902, 0.852)$	unstable
	$EQ_1 = (522.72, 21.69, 8.68, 1.39, 0.91, 0.09, 13.04, 0, 0)$	$(0.999, 0.999, 0.998, 0.998, 0.997, 0.990, 0.979, 0.923, 0.841)$	LAS
(III)	$EQ_0 = (1000, 0, 0, 3.20, 0, 0, 0, 0, 0)$	$(1.002, 0.999, 0.999, 0.998, 0.997, 0.990, 0.974, 0.927, 0.840)$	unstable
	$EQ_2 = (592.59, 0, 0, 3.198, 0, 0, 0, 101.85, 15.28)$	$(0.999, 0.999, 0.999, 0.9988, 0.996, 0.990, 0.976, 0.936, 0.837)$	LAS
(IV)	$EQ_0 = (1000, 0, 0, 3.20, 0, 0, 0, 0, 0)$	$(1.006, 1.002, 0.999, 0.999, 0.998, 0.990, 0.975, 0.912, 0.846)$	unstable
	$EQ_1 = (675.48, 14.75, 5.9, 0.16, 1.54, 0.15, 8.9, 0, 0)$	$(1\ , 0.999, 0.999, 0.998, 0.990, 0.976, 0.973, 0.923, 0.841)$	unstable
	$EQ_2 = (666.67, 0, 0, 3.20, 0, 0, 0, 83.33, 12.50)$	$(1.001, 0.999, 0.999, 0.999, 0.996, 0.990, 0.976, 0.924, 0.841)$	unstable
	$EQ_3 = (666.67, 3.77, 1.51, 0.4, 1.4, 0.14, 2.31, 62.59, 9.39)$	$(0.9999, 0.999, 0.999, 0.998, 0.992, 0.990, 0.976, 0.924, 0.841)$	LAS

6.2. Impact of Latent Reservoirs on the HIV-1 and HTLV-I Co-Dynamics

In this part, we investigate the impact of the presence of latent reservoirs on HIV-1 and HTLV-I co-dynamics. Currently, there is no available treatment for HTLV-I [61]. Therefore, we only consider one type of HIV-1 antiviral drug, reverse transcriptase inhibitor (RTI), which prevents the HIV-1 from infecting the macrophages and $CD4^+T$ cells. Let $\epsilon \in [0,1]$ be the efficacy of the RTI drug. Now, we write model (1) under the effect of RTI:

$$\begin{cases} \frac{dx}{dt} = \zeta_1 - \varrho_1 x - (1-\epsilon)\rho_1 x v - \rho_3 x u, \\ \frac{dy}{dt} = (1-\epsilon)\rho_1 x v - \alpha_1 y, \\ \frac{dw}{dt} = \zeta_2 - \varrho_2 w - (1-\epsilon)\rho_2 w v, \\ \frac{dz}{dt} = (1-\epsilon)\rho_2 w v - \alpha_2 z, \\ \frac{dv}{dt} = \beta_1\alpha_1 y + \beta_2\alpha_2 w - \theta v, \\ \frac{du}{dt} = \rho_3 x u - \delta u. \end{cases} \tag{59}$$

The basic reproductive numbers of model (59) are given by

$$R_0^{\text{Without latent}}(\epsilon) = \frac{(1-\epsilon)\beta_1\rho_1\zeta_1}{\theta\varrho_1} + \frac{(1-\epsilon)\beta_2\rho_2\zeta_2}{\theta\varrho_2},$$
$$R_1^{\text{Without latent}} = \frac{\rho_3\zeta_1}{\varrho_1\delta}.$$

Similarly, the model with latent (2) under the influence of RTI drug is given as:

$$\begin{cases} \frac{dx}{dt} = \zeta_1 - \varrho_1 x - (1-\epsilon)\rho_1 x v - \rho_3 x u, \\ \frac{dg}{dt} = (1-\epsilon)\rho_1 x v - (\pi_1+\mu_1) g, \\ \frac{dy}{dt} = \pi_1 g - \alpha_1 y, \\ \frac{dw}{dt} = \zeta_2 - \varrho_2 w - (1-\epsilon)\rho_2 w v, \\ \frac{ds}{dt} = (1-\epsilon)\rho_2 w v - (\pi_2+\mu_2) s, \\ \frac{dz}{dt} = \pi_2 s - \alpha_2 z, \\ \frac{dv}{dt} = \beta_1\alpha_1 y + \beta_2\alpha_2 z - \theta v, \\ \frac{dq}{dt} = \rho_3 x u - (\pi_3+\mu_3) q, \\ \frac{du}{dt} = \pi_3 q - \delta u. \end{cases} \tag{60}$$

The basic reproductive numbers of model (60) are given by

$$R_0^{\text{With latent}}(\epsilon) = \frac{(1-\epsilon)\beta_1\rho_1\pi_1\zeta_1}{\theta\varrho_1(\pi_1+\mu_1)} + \frac{(1-\epsilon)\beta_2\rho_2\pi_2\zeta_2}{\theta\varrho_2(\pi_2+\mu_2)},$$
$$R_1^{\text{With latent}} = \frac{\rho_3\zeta_1\pi_3}{\varrho_1\delta(\pi_3+\mu_3)}.$$

We suppose that $R_1^{\text{Without latent}} \leq 1$ and $R_1^{\text{With latent}} \leq 1$. In order to stabilize the infection-free equilibrium EQ_0 for both systems (59) and (60), we determine the minimum drug efficacies $\epsilon_{\min}^{\text{Without latent}}$ and $\epsilon_{\min}^{\text{With latent}}$ that make

$$R_0^{\text{Without latent}}(\epsilon) \leq 1, \text{ for all } \epsilon_{\min}^{\text{Without latent}} \leq \epsilon \leq 1,$$
$$R_0^{\text{With latent}}(\epsilon) \leq 1, \text{ for all } \epsilon_{\min}^{\text{With latent}} \leq \epsilon \leq 1,$$

where

$$\epsilon_{\min}^{\text{Without latent}} = \max\left\{0, 1 - \left(\frac{\beta_1\rho_1\zeta_1}{\theta\varrho_1} + \frac{\beta_2\rho_2\zeta_2}{\theta\varrho_2}\right)^{-1}\right\},$$
$$\epsilon_{\min}^{\text{With latent}} = \max\left\{0, 1 - \left(\frac{\beta_1\rho_1\pi_1\zeta_1}{\theta\varrho_1(\pi_1+\mu_1)} + \frac{\beta_2\rho_2\pi_2\zeta_2}{\theta\varrho_2(\pi_2+\mu_2)}\right)^{-1}\right\}.$$

We note that $R_0^{\text{With latent}} \leq R_0^{\text{Without latent}}$ and $\epsilon_{\min}^{\text{With latent}} \leq \epsilon_{\min}^{\text{Without latent}}$. This means that neglecting the latent reservoirs in the HIV-1 and HTLV-I coinfection model will lead to an overestimation of the required HIV-1 antiviral drugs.

Lengthening of the Latent Phase

In this part, we study the effect of lengthening the latent phase on the HIV-1 and HTLV-I co-dynamics. The average length of the latent phases are given by $1/\pi_i$, $i = 1,2,3$. Therefore, reducing the parameter π_i will increase the transition time from latent to active and then will slow the viral replication. Let us assume that there exist two control actions ϵ and $\bar{\epsilon}$ with the objective of reducing the transition rate (lengthening of the latent durations). These control actions may represent the efficacies of two drugs [62,63]. Let us multiply π_1 by $(1-\epsilon)$, π_2 by $(1-\epsilon)$ and π_3 by $(1-\bar{\epsilon})$, where $\epsilon, \bar{\epsilon} \in [0,1]$. Then, model (2) under the effect of such control actions can be written as:

$$\begin{cases} \frac{dx}{dt} = \zeta_1 - \varrho_1 x - \rho_1 xv - \rho_3 xu, \\ \frac{dg}{dt} = \rho_1 xv - ((1-\epsilon)\pi_1 + \mu_1)g, \\ \frac{dy}{dt} = (1-\epsilon)\pi_1 g - \alpha_1 y, \\ \frac{dw}{dt} = \zeta_2 - \varrho_2 w - \rho_2 wv, \\ \frac{ds}{dt} = \rho_2 wv - ((1-\epsilon)\pi_2 + \mu_2)s, \\ \frac{dz}{dt} = (1-\epsilon)\pi_2 s - \alpha_2 z, \\ \frac{dv}{dt} = \beta_1\alpha_1 y + \beta_2\alpha_2 z - \theta v, \\ \frac{dq}{dt} = \rho_3 xu - ((1-\bar{\epsilon})\pi_3 + \mu_3)q, \\ \frac{du}{dt} = (1-\bar{\epsilon})\pi_3 q - \delta u. \end{cases} \tag{61}$$

Let the model's parameters be fixed other than ϵ and $\bar{\epsilon}$, then the basic reproductive numbers of system (61) R_0 and R_1 as functions of ϵ and $\bar{\epsilon}$, respectively, are given by

$$R_0(\epsilon) = \frac{(1-\epsilon)\beta_1\rho_1\pi_1\zeta_1}{\theta\varrho_1((1-\epsilon)\pi_1+\mu_1)} + \frac{(1-\epsilon)\beta_2\rho_2\pi_2\zeta_2}{\theta\varrho_2((1-\epsilon)\pi_2+\mu_2)},$$
$$R_1(\bar{\epsilon}) = \frac{(1-\bar{\epsilon})\rho_3\zeta_1\pi_3}{\varrho_1\delta((1-\bar{\epsilon})\pi_3+\mu_3)},$$

and we have

$$\frac{dR_0}{d\epsilon} = -\frac{\beta_1 \mu_1 \zeta_1 \pi_1 \rho_1}{\varrho_1 \theta((1-\epsilon)\pi_1 + \mu_1)^2} - \frac{\beta_2 \mu_2 \zeta_2 \pi_2 \rho_2}{\varrho_2 \theta((1-\epsilon)\pi_2 + \mu_2)^2} < 0,$$
$$\frac{dR_1}{d\bar{\epsilon}} = -\frac{\mu_3 \zeta_1 \pi_3 \rho_3}{\varrho_1 \delta((1-\bar{\epsilon})\pi_3 + \mu_3)^2} < 0.$$

Hence, $R_0(\epsilon)$ and $R_1(\bar{\epsilon})$ are strictly decreasing functions of ϵ and $\bar{\epsilon}$, respectively. Therefore, increasing the values of ϵ and $\bar{\epsilon}$ will decrease the parameters R_0 and R_1. Now we determine the minimum control actions $\epsilon_{\min}$ and $\bar{\epsilon}_{\min}$ that make

$$R_0(\epsilon) \leq 1, \text{ for all } \epsilon_{\min} \leq \epsilon \leq 1,$$
$$R_1(\bar{\epsilon}) \leq 1, \text{ for all } \bar{\epsilon}_{\min} \leq \bar{\epsilon} \leq 1,$$

and stabilize the system around the infection-free equilibrium EQ_0. Using the values of the parameters given in Case (IV), we obtain $\epsilon_{\min} = 0.853353$ and $\bar{\epsilon}_{\min} = 0.666667$. Hence, both HIV-1 and HTLV-I will die out if the control actions satisfy $0.853353 \leq \epsilon \leq 1$ and $0.666667 \leq \bar{\epsilon} \leq 1$. It means that increasing the values of ϵ and $\bar{\epsilon}$ will lengthen the latent phase and then clear the viruses from the body. This give some impression to develop new drug therapies for lengthening the latent phase.

7. Discussion and Conclusions

In this paper, we developed and addressed a mathematical model that describes in-host co-dynamics of HIV-1 and HTLV-I with latent reservoirs. The nonlinear continuous-time model is discretized by using the NSFD method. We first proved the positivity and boundedness of the discrete-time model, and then we calculated the model's equilibria. We found that the model has four equilibria—infection-free equilibrium EQ_0, chronic HIV-1 single-infection equilibrium EQ_1, chronic HTLV-I single-infection equilibrium EQ_2 and chronic HIV-1/HTLV-I coinfection equilibrium EQ_3. We deduced the conditions that determine the existence and stability of equilibria. These conditions were given in terms of four threshold parameters $R_j > 0$, $j = 0, 1, 2, 3$. The global stability of all equilibria of the discrete-time model was examined by constructing Lyapunov functions. We proved that

- EQ_0 always exists, further, if $R_0 \leq 1$ and $R_1 \leq 1$, then EQ_0 is GAS. This result recommends that when $R_0 \leq 1$ and $R_1 \leq 1$, both HIV-1 and HTLV-I infections will be cleared regardless of the starting points.
- EQ_1 exists when $R_0 > 1$, and it is GAS when $R_0 > 1$ and $R_1 \leq 1$. This result recommends obtain when $R_0 > 1$ and $R_1 \leq 1$, the HIV-1 single-infection is always established regardless of the starting points.
- EQ_2 exists when $R_1 > 1$, and it is GAS when $R_1 > 1$ and $R_{02} + \frac{R_{01}}{R_1} \leq 1$. This result recommends that when $R_1 > 1$ and $R_{02} + \frac{R_{01}}{R_1} \leq 1$, the HTLV-I single-infection is always established regardless of the starting points.
- EQ_3 exists, and it is GAS when $\frac{R_1}{R_{01}} > 1$, $R_2 > 1$ and $R_3 > 1$. This result recommends that the HIV-1 and HTLV-I coinfection is always established regardless of the starting points.

We simulated the discrete-time model to confirm the theoretical results. We discussed the impact of latent reservoirs on the HIV-1 and HTLV-I co-dynamics. We established that including the latent reservoirs in the HIV-1 and HTLV-I coinfection model will reduce both R_0 and R_1. Therefore, neglecting the latent reservoirs can lead to an overestimation of the required HIV-1 antiviral drugs. Further, we found that lengthening the latent phase can suppress the viral progression. This may draw the attention of scientists and pharmaceutical companies to create new treatments that lengthen the latency period.

The HIV-1 and HTLV-I coinfection model presented in this work is given by a system of ordinary differential equations. The HIV-1 and HTLV-I co-dynamics can be described by (i) delay differential equations by incorporating the time delay [21], (ii) partial differential equations by considering the mobility of cells and viruses [64], (iii) stochastic differential equations by taking into account the random fluctuations [65–68] and (iv) fractional differential equations by considering the memory effects [69]. Great efforts are needed to study such points; therefore, we leave them for future work.

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Abbreviations

The following abbreviations are used in this manuscript:

Abbreviation	**Definition**
AIDS	Acquired immunodeficiency syndrome
ATL	Adult T-cell leukemia
CTLs	Cytotoxic T lymphocytes
GAS	Globally asymptotically stable
HAM/TSP	HTLV-I-associated myelopathy/tropical spastic paraparesis
HBV	Hepatitis B virus
HCV	Hepatitis C virus
HIV-1	Human immunodeficiency virus type 1
HTLV-I	Human T-lymphotropic virus type I
LAS	Locally asymptotically stable
NSFD	Non-standard finite difference
IAV	Influenza A virus
RTI	Reverse transcriptase inhibitor
SARS-CoV-2	Severe acute respiratory syndrome coronavirus 2
sup	Supremum (least upper bound)

References

1. Nowak, M.A.; Bangham, C.R.M. Population dynamics of immune responses to persistent viruses. *Science* **1996**, *272*, 74–79. [CrossRef]
2. Stilianakis, N.I.; Seydel, J. Modeling the T-cell dynamics and pathogenesis of HTLV-I infection. *Bull. Math. Biol.* **1999**, *61*, 935–947. [CrossRef] [PubMed]
3. Ciupe, S.M.; Ribeiro, R.M.; Nelson, P.W.; Perelson, A.S. Modeling the mechanisms of acute hepatitis B virus infection. *J. Theor. Biol.* **2007**, *247*, 23–35. [CrossRef] [PubMed]
4. Kitagawa, K.; Kuniya, T.; Nakaoka, S.; Asai, Y.; Watashi, K.; Iwami, S. Mathematical analysis of a transformed ODE from a PDE multiscale model of hepatitis C virus infection. *Bull. Math. Biol.* **2019**, *81*, 1427–1441. [CrossRef]
5. Baccam, P.; Beauchemin, C.; Macken, C.A.; Hayden, F.G.; Perelson, A.S. Kinetics of influenza A virus infection in humans. *J. Virol.* **2006**, *80*, 7590–7599. [CrossRef]
6. Hernandez-Vargas, E.A.; Velasco-Hernandez, J.X. In-host mathematical modelling of COVID-19 in humans. *Annu. Rev. Control* **2020**, *50*, 448–456. [CrossRef] [PubMed]
7. Wang, S.; Pan, Y.; Wang, Q.; Miao, H.; Brown, A.N.; Rong, L. Modeling the viral dynamics of SARS-CoV-2 infection. *Math. Biosci.* **2020**, *328*, 108438. [CrossRef]
8. Comez, M.C.; Yang, H.M. Mathematical model of the immune response to dengue virus. *J. Appl. Math. Comput.* **2020**, *63*, 455–478.

9. Wang, Y.; Liu, X. Stability and Hopf bifurcation of a within-host chikungunya virus infection model with two delays. *Math. Comput. Simul.* **2017**, *138*, 31–48. [CrossRef]
10. Best, K.; Perelson, A.S. Mathematical modeling of within-host Zika virus dynamics. *Immunol. Rev.* **2018**, *285*, 81–96. [CrossRef]
11. Tang, B.; Xiao, Y.; Sander, B.; Kulkarni, M.A.; RADAM-LAC Research Team; Wu, J. Modelling the impact of antibody-dependent enhancement on disease severity of Zika virus and dengue virus sequential and co-infection. *R. Soc. Open Sci.* **2020**, *7*, 191749. [CrossRef] [PubMed]
12. Elaiw, A.M.; AlShamrani, N.H. Analysis of a within-host HTLV-I/HIV-1 co-infection model with immunity. *Virus Res.* **2021**, *295*, 1–23. [CrossRef] [PubMed]
13. Elaiw, A.M.; AlShamrani, N.H. HTLV/HIV dual infection: Modeling and analysis. *Mathematics* **2021**, *9*, 51. [CrossRef]
14. Pinky, L.; Dobrovolny, H.M. SARS-CoV-2 coinfections: Could influenza and the common cold be beneficial? *J. Med. Virol.* **2020**, *92*, 2623–2630. [CrossRef] [PubMed]
15. Elaiw, A.M.; Alsulami, R.S.; Hobiny, A.D. Modeling and stability analysis of within-host IAV/SARS-CoV-2 coinfection with antibody immunity. *Mathematics* **2022**, *10*, 4382. [CrossRef]
16. Agha, A.D.A.; Elaiw, A.M.; Ramadan, S.A.A.E. Stability analysis of within-host SARS-CoV-2/HIV coinfection model. *Math. Methods Appl. Sci.* **2022**, *45*, 11403–11422. [CrossRef]
17. Elaiw, A.M.; Shflot, A.S.; Hobiny, A.D. Global stability of delayed SARS-CoV-2 and HTLV-I coinfection models within a host. *Mathematics* **2022**, *10*, 4756. [CrossRef]
18. Birger, R.; Kouyos, R.; Dushoff, J.; Grenfell, B. Modeling the effect of HIV coinfection on clearance and sustained virologic response during treatment for hepatitis C virus. *Epidemics* **2015**, *12*, 1–10. [CrossRef]
19. Nampala, H.; Livingstone, S.; Luboobi, L.; Mugisha, J.Y.T.; Obua, C.; Jablonska-Sabuka, M. Modelling hepatotoxicity and antiretroviral therapeutic effect in HIV/HBV coinfection. *Math. Biosci.* **2018**, *302*, 67–79. [CrossRef]
20. Nelson, P.W.; Murray, J.D.; Perelson, A.S. A model of HIV-1 pathogenesis that includes an intracellular delay. *Math. Biosci.* **2000**, *163*, 201–215. [CrossRef]
21. Nelson, P.W.; Perelson, A.S. Mathematical analysis of delay differential equation models of HIV-1 infection. *Math. Biosci.* **2002**, *179*, 73–94. [CrossRef]
22. Perelson, A.S.; Nelson, P.W. Mathematical analysis of HIV-1 dynamics in vivo. *SIAM Rev.* **1999**, *41*, 3–44. [CrossRef]
23. Lv, C.; Huang, L.; Yuan, Z. Global stability for an HIV-1 infection model with Beddington-DeAngelis incidence rate and CTL immune response. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 121–127. [CrossRef]
24. Lin, J.; Xu, R.; Tian, X. Threshold dynamics of an HIV-1 model with both viral and cellular infections, cell-mediated and humoral immune responses. *Math. Biosci. Eng.* **2018**, *16*, 292–319. [CrossRef] [PubMed]
25. Gao, Y.; Wang, J. Threshold dynamics of a delayed nonlocal reaction-diffusion HIV infection model with both cell-free and cell-to-cell transmissions. *J. Math. Anal. Appl.* **2020**, *488*, 124047. [CrossRef]
26. Feng, T.; Qiu, Z.; Meng, X.; Rong, L. Analysis of a stochastic HIV-1 infection model with degenerate diffusion. *Appl. Math. Comput.* **2019**, *348*, 437–455. [CrossRef]
27. Callaway, D.S.; Perelson, A.S. HIV-1 infection and low steady state viral loads. *Bull. Math. Biol.* **2002**, *64*, 29–64. [CrossRef]
28. Adams, B.M.; Banks, H.T.; Kwon, H.-D.; Tran, H.T. Dynamic multidrug therapies for HIV: Optimal and STI control approaches. *Math. Biosci. Eng.* **2004**, *1*, 223–241. [CrossRef] [PubMed]
29. Adams, B.M.; Banks, H.T.; Davidian, M.; Kwon, H.-D.; Tran, H.T.; Wynne, S.N.; Rosenberg, E.S. HIV dynamics: Modeling, data analysis, and optimal treatment protocols. *J. Comput. Appl. Math.* **2005**, *184*, 10–49. [CrossRef]
30. Rong, L.; Perelson, A.S. Modeling HIV persistence, the latent reservoir, and viral blips. *J. Theor. Biol.* **2009**, *260*, 308–331. [CrossRef] [PubMed]
31. Perelson, A.S.; Essunger, P.; Cao, Y.; Vesanen, M.; Hurley, A.; Saksela, K.; Markowitz, M.; Ho, D.D. Decay characteristics of HIV-1-infected compartments during combination therapy. *Nature* **1997**, *387*, 188–191. [CrossRef]
32. Korobeinikov, A. Global properties of basic virus dynamics models. *Bull. Math. Biol.* **2004**, *66*, 879–883. [CrossRef] [PubMed]
33. Lim, A.G.; Maini, P.K. HTLV-I infection: A dynamic struggle between viral persistence and host immunity. *J. Theor. Biol.* **2014**, *352*, 92–108. [CrossRef]
34. Wang, W.; Ma, W. Global dynamics of a reaction and diffusion model for an HTLV-I infection with mitotic division of actively infected cells. *J. Appl. Anal. Comput.* **2017**, *7*, 899–930. [CrossRef]
35. Li, F.; Ma, W. Dynamics analysis of an HTLV-1 infection model with mitotic division of actively infected cells and delayed CTL immune response. *Math. Methods Appl. Sci.* **2018**, *41*, 3000–3017. [CrossRef]
36. Khajanchi, S.; Bera, S.; Roy, T.K. Mathematical analysis of the global dynamics of a HTLV-I infection model, considering the role of cytotoxic T-lymphocytes. *Math. Comput. Simul.* **2021**, *180*, 354–378. [CrossRef]
37. Wang, Y.; Liu, J.; Heffernan, J.M. Viral dynamics of an HTLV-I infection model with intracellular delay and CTL immune response delay. *J. Math. Anal. Appl.* **2018**, *459*, 506–527. [CrossRef]
38. Elaiw, A.M.; AlShamrani, N.H.; Dahy, E.; Abdellatif, A. Stability of within host HTLV-I/HIV-1 co-infection in the presence of macrophages. *Int. J. Biomath.* **2022**, *16*, 2250066. [CrossRef]

39. Elaiw, A.M.; Aljahdali, A.K.; Hobiny, A.D. Dynamical [roperties of discrete-time HTLV-I and HIV-1 within-host coinfection model. *Axioms* **2023**, *12*, 201. [CrossRef]
40. Elaiw, A.M.; AlShamrani, N.H.; Dahy, E.; Abdellatif, A.A.; Raezah, A.A. Effect of macrophages and latent reservoirs on the dynamics of HTLV-I and HIV-1 coinfection. *Mathematics* **2023**, *11*, 592. [CrossRef]
41. Mickens, R.E. *Nonstandard Finite Difference Models of Differential Equations*; World Scientific: Singapore, 1994.
42. Korpusik, A. A nonstandard finite difference scheme for a basic model of cellular immune response to viral infection. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *43*, 369–384. [CrossRef]
43. Geng, Y.; Xu, J.; Hou, J. Discretization and dynamic consistency of a delayed and diffusive viral infection model. *Appl. Math. Comput.* **2018**, *316*, 282–295. [CrossRef]
44. Xu, J.; Geng, Y. Stabilty preserving NSFD scheme for a delayed viral infection model with cell-to-cell transmission and general nonlinear incidence. *J. Differ. Equ. Appl.* **2017**, *23*, 893–916. [CrossRef]
45. Manna, K. A non-standard finite difference scheme for a diffusive HBV infection model with capsids and time delay. *J. Differ. Equ. Appl.* **2017**, *23*, 1901–1911. [CrossRef]
46. Vaz, S.; Torres, D.F.M. Discrete-time system of an intracellular delayed HIV model with CTL immune response. *arXiv* **2022**, arXiv:2205.02199.
47. Salman, S.M. A nonstandard finite difference scheme and optimal control for an HIV model with Beddington-DeAngelis incidence and cure rate. *Eur. Phys. J. Plus* **2020**, *135*, 808. [CrossRef]
48. Liu, X.L.; Zhu, C.C. A non-standard finite difference scheme for a diffusive HIV-1 infection model with immune response and intracellular delay. *Axioms* **2022**, *11*, 129. [CrossRef]
49. Elaiw, A.M.; Alshaikh, M.A. Stability of discrete-time HIV dynamics models with three categories of infected $CD4^{+}$ T-cells. *Adv. Differ. Equ.* **2019**, *2019*, 407. [CrossRef]
50. Pasha, S.A.; Nawaz, Y.; Arif, M.S. On the nonstandard finite difference method for reaction–diffusion models. *Chaos Solitons Fractals* **2023**, *166*, 112929. [CrossRef]
51. Maamar, M.H.; Ehrhardt, M.; Tabharit, L. A Nonstandard Finite Difference Scheme for a Time-Fractional Model of Zika Virus Transmission. 2022. Available online: https://www.imacm.uni-wuppertal.de/fileadmin/imacm/preprints/2022/imacm_22_21.pdf (accessed on 15 January 2023).
52. Farooqi, A.; Ahmad, R.; Alotaibi, H.; Nofal, T.A.; Farooqi, R.; Khan, I. A comparative epidemiological stability analysis of predictor corrector type non-standard finite difference scheme for the transmissibility of measles. *Results Phys.* **2021**, *21*, 103756. [CrossRef]
53. Shi, P.; Dong, L. Dynamical behaviors of a discrete HIV-1 virus model with bilinear infective rate. *Math. Methods Appl. Sci.* **2014**, *37*, 2271–2280. [CrossRef]
54. Perelson, A.S.; Kirschner, D.E.; de Boer, R. Dynamics of HIV Infection of CD4+ T cells. *Math. Biosci.* **1993**, *114*, 81–125. [CrossRef] [PubMed]
55. Elaiw, A.M.; Raezah, A.A.; Azoz, S.A. Stability of delayed HIV dynamics models with two latent reservoirs and immune impairment. *Adv. Differ. Equ.* **2018**, *50*, 1–25. [CrossRef]
56. Mohri, H.; Bonhoeffer, S.; Monard, S.; Perelson, A.S.; Ho, D. Rapid turnover of T lymphocytes in SIV-infected rhesus macaques. *Science* **1998**, *279*, 1223–1227. [CrossRef] [PubMed]
57. Wodarz, D.; Nowak, M. Immune responses and viral phenotype: Do replication rate and cytopathogenicity influence virus load? *J. Theor. Med.* **1999**, *2*, 113. [CrossRef]
58. Elaiw, A.M.; Alshaikh, M.A. Stability preserving NSFD scheme for a general virus dynamics model with antibody and cell-mediated responses. *Chaos Solitons Fractals* **2020**, *138*, 109862. [CrossRef]
59. Wang, Y.; Liu, J.; Liu, L. Viral dynamics of an HIV model with latent infection incorporating antiretroviral therapy. *Adv. Differ. Equ.* **2016**, *2016*, 225. [CrossRef]
60. Li, M.Y.; Lim, A.G. Modelling the role of Tax expression in HTLV-1 persisence in vivo. *Bull. Math. Biol.* **2011**, *73*, 3008–3029. [CrossRef] [PubMed]
61. Raza, M.T.; Mizan, S.; Yasmin, F.; Akash, A.S.; Shahik, S. Epitope-based universal vaccine for Human T-lymphotropic virus-1 (HTLV-1). *PLoS ONE* **2021**, *16*, e0248001. [CrossRef]
62. Beauchemin, A.A.C.; McSharry, J.J.; Drusano, G.L.; Nguyen, J.T.; Went, G.T.; Ribeiro, R.M.; Perelson, A.S. Modeling amantadine treatment of influenza A virus in vitro. *J. Theor. Biol.* **2008**, *254*, 439–451. [CrossRef]
63. Dobrovolny, H.M. Quantifying the effect of remdesivir in rhesus macaques infected with SARSCoV-2. *Virology* **2020**, *550*, 61–69. [CrossRef] [PubMed]
64. Bellomo, N.; Outada, N.; Soler, J.; Tao, Y.; Winkler, M. Chemotaxis and cross diffusion models in complex environments: Modeling towards a multiscale vision. *Math. Model. Methods Appl. Sci.* **2022**, *32*, 713–792. [CrossRef]
65. Ali, I.; Khan, S.U. Dynamics and simulations of stochastic COVID-19 epidemic model using Legendre spectral collocation method. *AIMS Math.* **2023**, *8*, 4220–4236. [CrossRef]
66. Qi, K.; Jiang, D.; Hayat, T.; Alsaedi, A. Virus dynamic behavior of a stochastic HIV/AIDS infection model including two kinds of target cell infections and CTL immune. *Math. Comput. Simulat.* **2021**, *188*, 548–570. [CrossRef]
67. Ali, I.; Khan, S.U. Asymptotic behavior of three connected stochastic delay neoclassical growth systems using spectral technique. *Mathematics* **2022**, *10*, 3639. [CrossRef]

68. Ali, I.; Khan, S.U. Threshold of stochastic SIRS epidemic model from infectious to susceptible class with saturated incidence rate using spectral method. *Symmetry* **2022**, *14*, 1838. [CrossRef]
69. Chatterjee, A.N.; Basir, F.A.; Almuqrin, M.A.; Mondal, J.; Khan, I. SARS-CoV-2 infection with lytic and nonlytic immune responses: A fractional order optimal control theoretical study. *Results Phys.* **2021**, *26*, 104260. [CrossRef] [PubMed]