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The "Golden" Number Theory and New Properties of Natural Numbers

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Abstract

The main purpose of the present article is to give a brief description of the "golden" number theory and new properties of natural numbers following from it, in particular, *Z*-property, *D*-property, Φ-code, *F*-code, *L*-code. These properties are of big theoretical interest for number theory and can be used in computer science.
The article is written in popular form and is intended for a wide circle of mathematicians (including mathematics students) and specialists in computer science, who are interested in the histories of mathematics and new ideas in the development of number theory and its applications in computer science.

Keywords: "golden" ratio; Fibonacci and Lucas numbers; Z-property; D-property; Φ*-code; F-code; L-code.*

1 Introduction

As is known [1], number theory is one of the oldest mathematical theories. As is emphasized in [1], number theory "*is a branch of pure mathematics, devoted primarily to the study of the integers. It is sometimes called "The Queen of Mathematics" because of its foundational place in the discipline.*"

Recently, the new ideas in number theory aroused in the Mathematics of Harmony [2]. In this regard, author's article [3], is the great interest.

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The main purpose of the present article is a popular and compact description of the concept of the "golden" number theory and following from it new surprising properties of natural numbers, described in the article [3]. Also, it should be noted that new ways for positional representation of numbers, described in the article, are directly related to computer science. Therefore, in the article there are obtained new mathematical results, which touch not only number theory, but also foundations of computer science.

2 The Geometric Definitions of a Number

2.1 Euclidean Definition

Euclidean number theory, stated in book VII of Euclid's *Elements*, begins from two definitions [4].

Definition 1. A unit is that by virtue of which of the things each of the things that exist is called one.

Definition 2. A number is a multitude composed of units.

As highlighted in the article [4], *"Euclid treats the unit, 1, separately from numbers, 2, 3, and so forth… The numbers in definition 2 are meant to be whole positive numbers greater than 1, and definition 1 is meant to define the unit as 1. The word "monad," derived directly from the Greek, is sometimes used instead of "unit."*

Euclidean number theory has geometric interpretation and begins from geometric definition of a natural number. Euclid considered all numbers as geometric segments and such approach led him to the following definition of a number. Suppose that we have the infinite number of the "units" of 1:

$$S = \{1,1,1,...\} \tag{1}$$

Euclid named them *"monads"* and he did not consider a *"monad"* as a number. It was simply the *beginning of all numbers.* Then we can define a natural number *N* as some geometric segment, which can be represented as the sum of the *"monads"* taken from (1), i.e.,

$$N = \underbrace{1+1+...+1}_{N} \cdot \tag{2}$$

In spite of limiting simplicity of the definition (2), it had played a great role in mathematics, in particular, in number theory. This definition underlies many important mathematical concepts, for example, the concept of the *prime* and *composed* numbers and the concept of *divisibility*, which are the main concepts of *elementary number theory*.

2.2 Constructive Definition of Real Numbers

But there are also other definitions of a number, in particular, a *real number*. The so-called *"constructive approach"* to the definition of a *real number* is known [5]. According to [5], "*the real numbers can be constructed as a completion of the rational numbers in such a way that a sequence defined by a decimal or binary expansion like (3; 3.1; 3.14; 3.141; 3.1415; …) converges to a unique real number, in this case π."*

We can use the binary system

$$A = \sum_i a_i 2^i \left(i = 0, \pm 1, \pm 2, \pm 3, ... \right) \left(a_i \in \{0,1\} \right) \tag{3}$$

for the constructive definition of real numbers. Then, the definition of a real number *A* according to (3) has the following geometric interpretation. Consider now an infinite set of the "binary" segments of the length 2^n, that is,

$$B = \{2^n\}, \tag{4}$$

where n = 0, ±1, ±2, ±3, …. Then all real numbers can be represented by the sums (3), which consist of the "binary" segments taken from (4).

Notice that the number of the terms of the sum (3) is always *finite* but *potentially unlimited*, that is, the definition (3) is a brilliant example of *the potential infinity concept,* used in the *constructive mathematics* [6].

Clearly, that the definition (3) determines on the numerical axis only a part of real numbers, which can be represented exactly by the sum (3). We name such numbers *constructive real numbers*. All other real numbers, which cannot be represented by the finite sum (3), are called *non-constructive real numbers*.

What numbers can be referred to the *non-constructive* numbers within the framework of the definition (3)? Clearly, that all irrational numbers, in particular, the main mathematical constants π and e, the number $\sqrt{2}$, the golden ratio are referred to the "non-constructive" numbers. But within the framework of the definition (3) some "rational" numbers (for example, 2/3, 3/7, etc.), which cannot be represented by the finite sum (3), also are referred to the "non-constructive" numbers.

Notice that though the definition (3) considerably limits the set of real numbers, this fact does not belittle his significance from the "practical," computing point of view. It is easy to prove, that any "non-constructive" real number can be represented by (3) approximately, and the quantization error Δ will decrease in the process of increasing the terms of the sum (3), however $\Delta \neq 0$ for all the "non-constructive" real numbers. In essence, in modern computers we use only the "constructive" numbers, given by (3), however we do not have any problem with the "non-constructive" numbers, because they can be represented in the form (3) with the approximation error Δ that strives to 0 potentially.

2.3 Newton's Definition of Real Numbers

Within of many millennia, mathematicians developed and précised the concept of a number. In the 17th century during the origin of modern science, in particular, modern mathematics, a number of methods of studying the "continuous" processes is developed and the concept of a real number again goes out on the foreground. Most clearly a new definition of this concept is given by Isaac Newton (1643 - 1727), one of the founders of mathematical analysis, in his *Arithmetica Universalis* (1707):

"We understand a number not as the set of units, but as the abstract ratio of some magnitude to other one of the same kind taken for the unit".

This formulation gives us the uniform definition of a real number, rational or irrational. If you consider now the *Euclidean definition* (2) from the point of *Newton's definition*, we can see that the "monad" of 1 in (2) plays a role of the unit. In the "binary" notation (3) the number 2, that is, the radix of numeral system, plays a role of the unit (or "monad").

3 Codes of the Golden *p*-proportions as a New Geometric Definition of a Real Number

3.1 Bergman's Number System with Irrational Base

In 1957 the American wunderkind George Bergman published the paper *A number system with an irrational base* [7] in the authoritative journal "*Mathematics Magazine."*

The following sum is called *Bergman's number system*:

$$A = \sum_i a_i \Phi^i \left(i = 0, \pm 1, \pm 2, \pm 3, \ldots \right), \tag{5}$$

where A is a real number, a_i is a binary numeral (0 or 1) of the i-th digit, Φ^i is the weight of the i-th digit, Φ is the base of Bergman's system (5).

Let us compare Bergman's system (5) to the binary system (3), which underlies modern computer technology. These notations have the common property: they use the binary numerals 0 and 1 for representation of numbers, that is, from this point of view they relates to the class of the binary systems. The abridged notations of the sums (3) and (5) have one and the same form:

$$A = a_n a_{n-1} \ldots a_1 a_0 . a_{-1} a_{-2} \ldots a_{-(k-1)} a_{-k} . \tag{6}$$

However, a principal distinction between them consists of the fact that the irrational number $\Phi = \frac{1+\sqrt{5}}{2}$ (the golden ratio) is used as the base of the numeral system (5) and the digit weights are connected by the following relations:

$$\Phi^i = \Phi^{i-1} + \Phi^{i-2} \tag{7}$$

$$\Phi^i = \Phi \times \Phi^{i-1}. \tag{8}$$

The first relation is called *additive* relation, the second - *multiplicative* relation.

For the case of Bergman's system (5), we will name the abridged representation (6) as the *"golden" representation*.

Bergman's system (5) is called a *number system with an irrational base*. Although Bergman's article [7] is the result of a principal importance for number theory and computer science, Bergman's article [7] in that period (1957) did be noted neither by mathematicians nor specialists in computer science. And in conclusion of his article [7] George Bergman wrote: *"I do not know of any useful application for systems such as this, except as a mental exercise and pastime, though it may be of some service in algebraic number theory."*

It is necessary to note that the number system with irrational base, developed by George Bergman in 1957, is the most important mathematical discovery of 20 c. in the field of numeral systems after the discovery of positional principle of number representation (Babylon, 2000 B.C.), decimal system (India, 5th century) and binary system (3). The most surprising is the fact that George Bergman made his mathematical discovery in the age of 12 years! This is an unprecedented case in the history of mathematics!

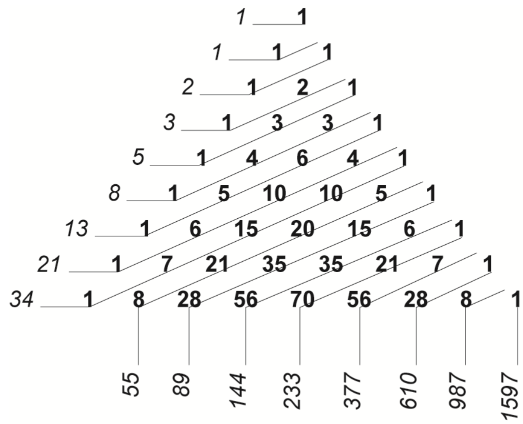
3.2 Pascal's Triangle and Fibonacci numbers

Binomial coefficients and Pascal's triangle is widely used in various fields of mathematics, computer science and many other applied sciences. Essentially, this is one of the fundamental mathematical notions that underlie the exact sciences. The famous mathematician Jacob Bernoulli (1655 - 1705) wrote:

"This table has a number of wonderful properties. We have just shown that this table is the essence of theory of combinations, but those who are in close contact with the geometry, know that this table contains a number of fundamental secrets of this area of mathematics".

In the book [8], the famous American mathematician and populariser of mathematics George Polya (1887 - 1985) has found a surprising connection between *Fibonacci numbers* and *"diagonal sums" of Pascal's triangle* (see Table 1) .

Table 1. Diagonal sums of Pascal's triangle



If we calculate the sums of binomial coefficients standing on the diagonals:

1=1, 1=1, 2=1+1, 3=1+2, 5=1+3+1, 8=1+4+3, 13=1+5+6+1, 21=1+6+10+1, …, we get the Fibonacci sequence, that is, Pascal's triangle is a "generator" of the Fibonacci numbers!

At first glance, it seems that finding this connection is so simple and so "elementary" what it hardly worthy the attention of mathematicians. However, this mathematical result, which, as say, "lay on the surface," for several centuries was the "big secret" for Blaise Pascal and other mathematicians who studied *Fibonacci numbers* and *Pascal's triangle*. However, the surprisingly simple mathematical connection between Fibonacci numbers and Pascal triangle opens the way to the deep union of two important mathematical theories, the *theory of Fibonacci numbers* [9 - 11] and *combinatorics* and this union can become a fruitful source for new mathematical ideas and generalizations.

3.3 Fibonacci *p*-numbers

The development of *Polya's idea* [8] led us in the book [12] to the discovery of the surprising generalized recurrence relation, which "generates" an infinite number of the new recurrence sequences, *Fibonacci p-numbers* (p=0,1,2,3,…) given by the following recurrence relation:

$$F_p(i) = F_p(i-1) + F_p(i-p-1) \quad \text{for} \quad i > p+1 \tag{9}$$

at the following seeds:

$$F_p(1) = F_p(2) = ... = F_p(p+1) = 1 \tag{10}$$

Here the numbers p=0,1,2,3,… correspond to the different inclinations of diagonals in Pascal's Triangle (Table 1).

Note that the recurrence relation (9) at the seeds (10) generates many remarkable numerical sequences, in particular, the *binary sequence* for the case p=0:

$$1,2,4,8,16,32,64,\ldots,2^{n-1},\ldots, \tag{11}$$

and the *Fibonacci sequence* for the case p=1:

$$1,1,2,3,5,8,13,21,34,55,\ldots, F_n,\ldots \tag{12}$$

3.4 The "Golden" *p*-proportions

A study of the limit of the ratios of two adjacent *Fibonacci p-numbers* led to the following algebraic equation:

$$x^{p+1} - x^p - 1 = 0, \quad (13)$$

which is a generalisation of the "golden" algebraic equation $x^2 - x - 1 = 0$ with the positive root $\Phi = \frac{1+\sqrt{5}}{2}$ (the golden ratio).

In general, the positive roots Φ_p of the equation (13) are new mathematical constants called the *golden p-proportions* (p=0,1,2,3,…) that are a generalization of the classical golden proportion [13, 14].

Table 2. Partial cases of Φ_p

p	0	1	2	3	4
Φ_p	2	1.618	1.4656	1.3802	1.3247

Thus, we have every right to claim that *Pascal's triangle* is a "generator" of new recurrent sequences $F_p(n)$, which are a generalization of the classical Fibonacci numbers, and new mathematical constants Φ_p, which are a generalization of the classical golden ratio. These mathematical concepts underlie the "*theory of the Fibonacci p-numbers and golden p-proportions,"* [12-14], which is a generalization of the classical *"theory of Fibonacci numbers and golden ratio*" [9-11].

3.5 Codes of the Golden *p*-proportions

The binary system (3) and Bergman's system (5) allow making the following generalization. Consider the set of the following standard line segments:

$$\{\ldots,\Phi_p^n,\Phi_p^{n-1},\ldots,\Phi_p^{n-p-1},\ldots,\Phi_p^0=1,\Phi_p^{-1},\ldots,\Phi_p^{-k},\ldots\} \quad (14)$$

where Φ_p is the golden p-proportion, a real root of the golden p-ratio equation (13). The powers of the golden p-proportions are connected by the remarkable identity:

$$\Phi_p^n = \Phi_p^{n-1} + \Phi_p^{n-p-1}. \quad (15)$$

By using (14), we can get the following positional binary method of real numbers representation called the *code of the golden p-proportions* [13,14]:

$$A = \sum_i a_i \Phi_p^i;\ p = 0,1,2,3,\ldots \quad (16)$$

where $a_i \in \{0,1\}$ is the binary numeral of the i-th digit; Φ_p^i is the weight of the i-th digit; Φ_p is the base of the numeral system (16), i = 0, ±1, ±2, ±3, … .

Note that the formula (16) gives an infinite number of new positional numeral systems with irrational bases, since all their bases $\Phi_p\left(p=1,2,3,...\right)$ are irrational numbers (except for the base $\Phi_{p=0}=2$, corresponding to the case p=0).

Note that the codes of the golden p-proportions (16) for the first time were introduced by Alexey Stakhov in 1980 in the article [13]. Theory of the codes of the golden p-proportions and their applications in computer science is described in Stakhov's 1984 book [14].

3.6 Surprising Properties of Natural Numbers

3.6.1 What means "elementary number theory"?

A detailed analysis of the history of *elementary number theory* and its applications is given in the book [15]:

"As far back as 5000 years ago, ancient civilizations had developed ways of expressing and doing arithmetic with integers. Throughout history, different methods have been used to denote integers. For instance, the ancient Babylonians used 60 as the base for their number system and the Mayans used 20. Our method of expressing integers, the decimal system, was first developed in India approximately six centuries ago. With the advent of modern computers, the binary system came into widespread use. Number theory has been used in many ways to devise algorithms for efficient computer arithmetic and for computer operations with large integers."

We can see from the above quote, that numeral systems similar to the *Babylonian system with the base 60, Mayan system with the base 20, decimal and binary systems* underlies *elementary number theory* and its main goal is to study integer numbers. In this connection, surprising properties of natural numbers, discovered in the "golden" number theory [3], are of theoretical interest for number theory and computer science.

3.6.2 Representation of natural numbers in Bergman's systems and codes of the golden *p*-proportions

We'll start from the simple examples of representation of natural numbers in Bergman's system (5). Let us consider the representation of natural number N in Bergman's system, called $\Phi-code$ of natural number N [3]:

$$\Phi-code: N=\sum_i a_i\Phi^i\left(i=0,\pm1,\pm2,\pm3,...\right). \quad (17)$$

The different binary "golden" representations (6) of one and the same natural number N for the $\Phi-code$ (17) can be obtained one from another by means of the peculiar code conversions called *convolution* and *devolution* of the binary digit. These code conversions are carried out within the scope of the one "golden" representation (6) and follow from the basic recurrence relation (7) that connects the adjacent digit weights of the $\Phi-code$ (17). The idea of such code conversions (micro-operations) consists of the following.

$$\begin{array}{l}\textbf{Convolution}: 011=100\\ \textbf{Devolution}:\ 100=011\end{array} \quad (18)$$

Below we see the examples of *convolutions* and *devolutions* for the natural numbers 7 and 5:

$$\textbf{Convolutions}: 7=\begin{cases}0\ 1\ 1\ 1\ 1\\ 1\ 0\ 0\ 1\ 1\\ 1\ 0\ 1\ 0\ 0\end{cases} \quad (19)$$

$$\textbf{Devolutions}:5=\begin{cases}1\ 0\ 0\ 0\ 0\\0\ 1\ 1\ 0\ 0\\0\ 1\ 0\ 1\ 1\end{cases} \tag{20}$$

The “golden” representation (6) for the number of 1 looks as follows:

$$1=\Phi^0=1.00 \tag{21}$$

By using the micro-operation of *devolution* [100=011], we obtain another “golden” representation of the number of 1 as follows:

$$1=0.11=\Phi^{-1}+\Phi^{-2} \tag{22}$$

Let us add the bit of 1 to the 0-th digit of the “golden” representation (22). As a result, we obtain the “golden” representation of the number of 2:

$$2=1.11 \tag{23}$$

If we carry out the micro-operation of *convolution* [011=100] to the “golden” representation (23), we obtain another “golden” representation of the number of 2:

$$2=10.01=\Phi^1+\Phi^{-2} \tag{24}$$

By adding the bit of 1 to the 0-th digit of the “golden” representation (24) after *convolution* [011=100], we obtain the following “golden” representation of the number of 3:

$$3=100.01=\Phi^2+\Phi^{-2} \tag{25}$$

The “golden” representation of the numbers 4 and 5 looks as follows:

$$\begin{aligned}4&=101.01=\Phi^2+\Phi^0+\Phi^{-2}\\5&=1000.1001=\Phi^3+\Phi^{-1}+\Phi^{-4}\end{aligned} \tag{26}$$

Continuing this process ad infinitum, we obtain the “golden” representations of all natural numbers. Thus, this study leads us to the following far not trivial mathematical result, which can be formulated as the following theorem.

Theorem 1. All natural numbers can be represented in Bergman’s system (17) by using a finite number of binary numerals.

This result can be generalized for the codes of the golden *p*-proportions [2].

Theorem 2. For the given $p>0$, any natural number can be represented in the golden *p*-proportion code (16) by using the finite number of bits.

Note that Theorems 1 and 2 are valid only for natural numbers. This gives us the right to interpret Theorem 1 and 2 as new properties of natural numbers.

3.7 The *Z*- and *D*-properties of Natural Numbers

Bergman's system (5) and codes of the golden *p*-proportions (16) are sources for new number-theoretical results. The *Z*- and *D*-properties are the most surprising among them.

Let us consider the $\Phi - code$ of natural number *N*, given by the formula (17). Note that according to Theorem 1, the sum (17) is always finite for any natural number *N*, that is, any natural number can be represented as a finite sum of the golden ratio powers.

Bergman's system (5) is a source of new number-theoretical results. The *Z*-property of natural numbers is one of these results. This property is based upon the following very simple reasoning.

There is well-known in Fibonacci numbers theory [11] the following remarkable formula:

$$\Phi^i = \frac{L_i + F_i\sqrt{5}}{2}\left(i = 0, \pm 1, \pm 2, \pm 3, \ldots\right), \tag{27}$$

which connects the golden ratio powers Φ^i through Lucas number L_i and Fibonacci numbers F_i.

Recall that in the formula (27) we are talking about Fibonacci and Lucas numbers, "extended" to the negative values of the index $i = 0, \pm 1, \pm 2, \pm 3, \ldots$. Table 3 demonstrates the example of the "extended" Fibonacci and Lucas numbers.

Table 3. The "extended" Fibonacci and Lucas numbers

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	-1	2	-3	5	-8	13	-21	34	-55
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	-1	3	-4	7	-11	18	-29	47	-76	123

It follows from Table 3 the following identities, which connect the "extended" Fibonacci and Lucas numbers:

$$F_{-n} = (-1)^{n+1} F_n; \quad L_{-n} = (-1)^n L_n \tag{28}$$

If we substitute the expression (27) instead Φ^i into the formula (5), we can represent the Φ-code (5) of natural number *N* as follows:

$$N = \frac{1}{2}\left(A + B\sqrt{5}\right), \tag{29}$$

where

$$A = \sum_i a_i L_i \tag{30}$$

$$B = \sum_i a_i F_i. \tag{31}$$

Note that all binary numerals {0,1} in the sums (30) and (31) coincide with the corresponding binary numerals of the Φ-code (5) of natural number *N*.

Let us represent the formula (29) as follows:

$$2N = A + B\sqrt{5}. \tag{32}$$

Note that the expression (32) has general character and is valid for any arbitrary natural number *N*.

Let us study the "strange" formula (32), which is valid for any natural number *N*. It is clear that the number 2*N* that stands on the left of the expression (32) is always an even number. The right-hand part of the expression (32) is the sum of the number *A* and the product of the number *B* by the irrational number $\sqrt{5}$. However, according to (30) and (31), the numbers *A* and *B* are always integer numbers because the Fibonacci and Lucas numbers are integers. Then, it follows from (32) that for any natural number *N*, the even number 2*N* is equal to the sum of the integer *A* and the product of the integer *B* by $\sqrt{5}$. This assertion is valid for all natural numbers N! We can ask the question: when the expression (32) is valid for the general case? The answer to this question is very simple: the expression (32) will be valid for any natural number *N* only for the condition if the sum (31) is equal to 0 ("zero"), and the sum (30) is equal to the doubled *N*, that is,

$$B = \sum_i a_i F_i = 0 \tag{33}$$

$$A = \sum_i a_i L_i = 2N. \tag{34}$$

Next let us compare the sums (17) and (31). Since the binary numerals a_i in these sums coincide, it follows that the expression (31) can be obtained from the formula (17) if we substitute the Fibonacci number F_i instead of every power of the golden ratio Φ^i in the formula (17), where the index *i* takes its values from the set {0,±1, ±2, ±3,…}. However, according to (33) the sum (31) is equal to 0 identically, independently of the initial natural number *N* in the formula (17). Thus, we have discovered a new fundamental property of natural numbers, which can be formulated through the following theorem.

Theorem 3 (*Z*-property). If we represent an arbitrary natural number *N* in the Φ-code (17) and then substitute the Fibonacci number F_i instead of the golden ratio power Φ^i in the expression (17), where the index *i* takes its values from the set {0,±1,±2,±3,…}, then the sum that appear as a result of such a substitution is equal to 0 identically, independently on the initial natural number *N*, that is,

$$\text{For any } N = \sum_i a_i \Phi^i \text{ after substitution } F_i \to \Phi^i : \sum_i a_i F_i \equiv 0 \left(i = 0, \pm 1, \pm 2, \pm 3, ...\right) \tag{35}$$

Let's compare now the sum (17) and (30). Since the binary numerals a_i in these sums coincide, the formula (30) can be obtained from the formula (17) if we substitute the Lucas number L_i instead of the golden ratio power Φ^i in the formula (17), where the index *i* takes its values from the set {0,±1,±2,±3,…}. However, according to (34) the sum (30) is equal to 2*N* identically, independently of the initial natural number *N* in the formula (17). Thus, we have discovered another fundamental property of natural numbers, which can be formulated as the following theorem.

Theorem 4 (*D*-property). If we represent an arbitrary natural number *N* in the Φ-code (17) and then substitute the Lucas number L_i instead of the golden ratio power Φ^i in the formula (17), where the index *i* takes its values from the set {0,±1,±2,±3,…}, then the sum that appears as a result of such a substitution is equal to 2*N* identically, independently of the initial natural number *N*, that is,

$$\textit{For any } N=\sum_i a_i\Phi^i \textit{ after substitution } F_i \to \Phi^i : \sum_i a_i L_i \equiv 2N\left(i=0,\pm1,\pm2,\pm3,...\right). \quad (36)$$

Thus, Theorems 3 and 4 provide new fundamental properties of natural numbers. For the first time the *Z*- and *D*-properties of natural numbers are described in author's article [3] published in the *Ukrainian Mathematical Journal.* It is surprising for many mathematicians to know that the new mathematical properties of natural numbers were only discovered at the beginning of the 21-th century, that is, 2½ millennia after the beginning of their theoretical study. The golden ratio and the "extended" Fibonacci and Lucas numbers play a fundamental role in this discovery. This discovery connects together two outstanding mathematical concepts of Greek mathematics - *Natural Numbers* and the *Golden Section.* This discovery is the second confirmation of the fruitfulness of the constructive approach to the number theory based upon Bergman's system (5).

3.8 The *F*- and *L*-codes

3.8.1 Definition of the *F*- and *L*-codes

The above *Z*- and *D*-properties of natural numbers given by Theorems 3 and 4, allow us to create new and very unusual codes for the representation of natural numbers.

Taking into consideration the *Z*-property (35), we can rewrite the expression (29) as follows:

$$N=\frac{1}{2}\left(A+B\right), \quad (37)$$

where *A* is defined by the expression (30) and *B* by the expression (31).

Let us explain the expression (37). According to the formula (31), $B=\sum_i a_i F_i=0$ (Z-property). Therefore, the formula (37) can be rewritten as follows: $N=\frac{1}{2}\left(A+0\right)$. On the other hand, we can rewrite the formula (37) as follows: $N=\frac{1}{2}\left(\sum_i a_i L_i+\sum_i a_i F_i\right)=\frac{1}{2}\left[\sum_i a_i\left(L_i+F_i\right)\right]$, because all binary coefficients in the expressions (30), (31) coincide. It follows from these arguments the following expression for *N*:

$$N=\sum_i a_i\frac{L_i+F_i}{2}\left(i=0,\pm1,\pm2,\pm3,...\right). \quad (38)$$

Taking into consideration the following well-known identity [11]

$$\frac{L_i+F_i}{2}=F_{i+1},$$

we get from (38) the following representation of the same natural number *N*:

$$N=\sum_i a_i F_{i+1}\left(i=0,\pm1,\pm2,\pm3,...\right). \quad (39)$$

The expression (39) is named the *F-code of natural number N* [3].

As the binary numerals $a_i \in \{0,1\}$ in the expressions (17) and (39) coincide, it follows from this fact that the *F*-code of the natural number *N* can be obtained from the Φ-code (17) of the same natural number *N* by means of substitution of the Fibonacci number F_{i+1} instead of the golden ratio power Φ^i, where i=0,±1,±2,±3,….

Let us now represent the *F*-code of *N*, given by (39), as follows:

$$N = \sum_i a_i F_{i+1} + 2B, \tag{40}$$

where the term *B* is defined by the expression (31). Note that according to (35) the expression (31) is equal to 0 identically. Then, the expression (40) can be represented as follows:

$$N = \sum_i a_i (F_{i+1} + 2F_i). \tag{41}$$

Taking into consideration the following well known identity [11]

$$L_{i+1}=F_{i+1}+2F_i,$$

the expression (41) can be represented as follows:

$$N = \sum_i a_i L_{i+1} \left(i = 0, \pm 1, \pm 2, \pm 3, \ldots \right). \tag{42}$$

The expression (42) is named the *L-code of natural number N* [3].

As the binary numerals $a_i \in \{0,1\}$ in the expressions (17) and (42) coincide, this means that the *L*-code of *N*, given by (42), can be obtained from the Φ-code of *N* (17) by means of the substitution of the Lucas numbers L_{i+1} instead of the golden ratio powers Φ^i, where i=0,±1,±2,±3,… . It is clear that the *L*-code of *N*, given by (42), can also be obtained from the *F*-code (39) of the same number *N* by means of the substitution of the Lucas number L_{i+1} instead of the Fibonacci number F_{i+1} in the formula (39).

Let us consider the representation of the sums (17), (39) and (42) in the abridged form (6) called the "golden" representation of natural number *N*. It is clear that the Φ-code (17), *F*-code (39) and *L*-code (42) lead us to one and the same "golden" representation of natural number *N* in the form (6) because all the binary numerals $a_i \in \{0,1\}$ in the expressions (17), (39) and (42) coincide.

3.8.2 Numerical example

Once again let us consider the "golden" representation (6). We can see that the "golden" representation (6) is divided by the point into two parts, namely the left and the right hand parts, which respectively consist of the digits with non-negative and negative indices. For example, let us consider the "golden" representation of the decimal number 10 in Bergman's system:

$$10=10100.0101. \tag{43}$$

For the Φ-code (17) the "golden" representation (43) has the following numerical interpretation:

$$10 = \Phi^4 + \Phi^2 + \Phi^{-2} + \Phi^{-4} \tag{44}$$

By using the formula (28), we can represent the sum (44) as follows:

$$10 = \frac{L_4 + F_4\sqrt{5}}{2} + \frac{L_2 + F_2\sqrt{5}}{2} + \frac{L_{-2} + F_{-2}\sqrt{5}}{2} + \frac{L_{-4} + F_{-4}\sqrt{5}}{2}. \quad (45)$$

From the relation (28), we can easily obtain:

$$L_{-2} = L_2, L_{-4} = L_4, F_{-2} = -F_2, F_{-4} = F_4 \quad (46)$$

Then (45) can be reduced as follows:

$$10 = \frac{2(L_4 + L_2)}{2} = L_4 + L_2 = 7 + 3.$$

Now, let us consider the interpretation of the "golden" representation (6) as the *F*- and *L*-codes:

$10=F_5+F_3+F_{-1}+F_{-3}=5+2+1+2$;

$10=L_5+L_3+L_{-1}+L_{-3}=11+4-1-4$.

Also we can check the sum (45) according to the *Z*- and *D*-properties. If we substitute into (45) the "extended" Fibonacci (F_i) and Lucas (L_i) numbers instead of the powers Φ^i, we get the following sums:

$F_4+F_2+F_{-2}+F_{-4}=3+1+(-1)+(-4)=0$ (*Z*-property)

$L_4+L_2+L_{-2}+L_{-4}=7+3+3+7=20=2\times10$ (*D*-property).

Thus, there are three way of the interpretation of the "golden" representation (6) of one and the same natural number *N*: $\Phi - code$ (17), *F-code* (39) and *L-code* (42). Wherein, as is shown in [3], the distinction between them arises only at shifting the "golden" representation (6) to the left or to the right.

3.8.3 Shifting the *F*- and *L*-codes

Let us denote by $N_{(k)}$ and $N_{(-k)}$ the results of shifting the "golden" representation (6) on the *k* digits to the left and to the right, respectively.

The following theorems are proved in [3].

Theorem 5. Shifting the "golden" representation (6), interpreted as the *F*-code (39), on the *k* digits to the left (that is, to the side of the highest digits) corresponds to the multiplication of the number *N* by the Fibonacci number F_{k+1}, however, its shifting on the *k* digits to the right (that is, to the side of the lowest digits) corresponds to the multiplication of the number *N* by the Fibonacci number F_{-k+1}, that is,

$$N_{(+k)} = \sum_i a_i F_{i+k+1} = F_{k+1} \times N.. \quad (47)$$

$$N_{(-k)} = \sum_i a_i F_{i-k+1} = F_{-k+1} \times N. \quad (48)$$

Theorem 6. Shifting the "golden" representation (6), interpreted as the *L*-code (42), on the *k* digits to the left (that is, to the side of the highest digits) corresponds to the multiplication of the number *N* by the Lucas number L_{k+1}, however, its shifting on the *k* digits to the right (that is, to the side of the lowest digits) corresponds to the multiplication of the number *N* by the Lucas number L_{-k+1}, that is,

$$N_{(k)} = \sum_i a_i L_{i+k+1} = L_{k+1} \times N \quad (49)$$

$$N_{(-k)} = \sum_i a_i L_{i-k+1} = L_{-k+1} \times N. \quad (50)$$

Note that the theorems, similar to Theorems 3 – 6, are proved for the codes of the golden *p*-proportions in the book [2].

4 Numeral Systems with Irrational Bases as the Basis of the "Golden" Number Theory

Above we developed the so-called "constructive" approach to the definition of real number based on the binary system (3). This idea allows making the following generalization. We can apply Newton's definition of real number to the numeral systems with irrational bases, given by (5),(16). In fact, the systems with irrational bases are fundamentally new numeral systems that have theoretical importance for number theory. They turn over our ideas about real numbers and their representation. Historically natural numbers were the first class of real numbers; the irrational numbers have been introduced into mathematics much later, after the discovery of the "incommensurable segments." In the traditional systems (Babylonian sexagecimal, decimal, binary) some natural numbers 60, 10, 2 are used as the "beginning of calculus." All real numbers can be represented through the bases 60, 10 or 2 by using sexagecimal, decimal, binary systems. In the systems with irrational bases (5), (16) some irrational numbers $\Phi = \frac{1+\sqrt{5}}{2}$ (the golden ratio) and Φ_p (the golden p-proportions) are the "beginning of calculus." All other real numbers (including natural numbers) can be represented through the irrational numbers Φ and $\Phi_p \left(p = 1,2,3,...\right)$ by using (5), (16).

Thus, in our concept we consider the sums (5), (16) as a new definition of real numbers. Above we have considered the extreme partial cases of the number representation (16). For the case *p*=0 the formula (16) is reduced to the binary system (3), for the case *p*=1 the system (16) is reduced to Bergman's system (5). It follows from this consideration that the positional representation (16) is very wide generalization of the classical binary system (2) and Bergman's system (5).

5 Conclusion

In conclusion, we note that all Theorems 1 – 6, formulated above, are valid only for natural numbers. Therefore, we have a right to consider the results of Theorems 1 – 6 as new properties of natural numbers. This means that in the "Harmony Mathematics" [2] we found unexpectedly new, previously unknown properties of natural numbers, the theoretical study of which began 2.5 millennia ago, at least starting from Euclid's *Elements*. These properties are of great interest for number theory and can be used in computer science.

Competing Interests

Author has declared that no competing interests exist.

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