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k-*Paranormal, k-Quasi-*paranormal and (n, k)- Quasi-*paranormal Composite Multiplication Operator on L^2-spaces

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Abstract

An operator $A \in B(H)$, A is said to be (n,k)-quasi-*paranormal if $\left\| A^{1+n}(A^k(x)) \right\|^{\frac{1}{1+n}} \left\| A^k(x) \right\|^{\frac{n}{1+n}} \geq \left\| A^*(A^k(x)) \right\|$ for every x in H[1]. In this paper, the conditions under which composite multiplication operator becomes k-*paranormal operator, k-quasi-*paranormal operator and (n,k) -quasi-*paranormal operator, have been obtained in terms of Radon-Nikodym derivative f_0.

*Keywords: k-*paranormal; k-quasi-*paranormal;* (n,k) *-quasi-*paranormal; conditional expectation; composition operator; multiplication operator and composite multiplication operator.*

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1 Introduction

Let X be a non-empty set, C be the field of complex numbers and $V(X)$ be a vector space of complex valued functions on X under the pointwise operations of addition and scalar multiplication. Let T be a

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mapping of X into X such that $f \circ T$ is in $V(X)$ whenever f is in $V(X)$. Define the composition transformation C_T on $V(X)$ as $C_T f = f \circ T$ for every f in $V(X)$. If $V(X)$ has a Banach space structure and C_T is bounded, then C_T is called the composition operator on $V(X)$ induced by T. Let $u: X \to C$ be a function such that M_u, defined as $M_u f = u \cdot f$ for every f in $V(X)$ is a bounded linear operator on $V(X)$. Then the product $C_T M_u$ which becomes a bounded operator on $V(X)$ is called a composite multiplication operator.

Let $B(H)$ be the Banach algebra of all bounded operators on a Hilbert Space H. If (X, Σ, μ) is a σ-finite measure space an $T: X \to X$ is a measurable transformation such that $C_T \in B(L^2(\mu))$, then in [2] R. K. Singh and D. C. Kumar have proved that the measure μT^{-1}, defined as $\mu T^{-1}(E) = \mu(T^{-1}(E))$ for every E in Σ, is absolutely continuous with respect to the measure μ. Let f_0 denote the Radon-Nikodym derivative of μT^{-1} with respect to μ and if $C_T \in B(L^2(\mu))$, then in [2] R. K. Singh has proved that $C_T^* C_T = M_{f_0}$. A composite multiplication operator is a linear transformation acting on a set of complex valued Σ measurable functions f of the form

$$M_{u,T}(f) = C_T M_u(f) = u \circ T \ f \circ T$$

Where u is a complex valued, Σ measurable function. In case $u = 1$ almost everywhere, $M_{u,T}$ becomes a composition operator, denoted by C_T.

Let X be a non-empty set and let Σ be a σ-algebra on X. Let μ and μT^{-1} be measures on Σ and $f_0 : X \to [0, \infty]$ be a measurable function, then the following are equivalent:

(i) μT^{-1} is absolutely continuous with respect to μ and f_0 is Radon-Nikodym derivative of μT^{-1} with respect to μ.

(ii) For every measurable function $f : X \to [0, \infty]$, the equality

$$\int_X f \, d\mu T^{-1} = \int_X f_0 \, f \, d\mu$$

holds.

In the study considered is the using conditional expectation of weighted composition operator on L^2-spaces. For each $f \in L^p(X, \Sigma, \mu)$, $1 \leq p \leq \infty$, there exists an unique $T^{-1}(\Sigma)$-measurable function $E(f)$ such that

$$\int_A g \, f \, d\mu = \int_A g \, E(f) \, d\mu$$

for every $T^{-1}(\Sigma)$-measurable function g, for which the left integral exists. The function $E(f)$ is called the conditional expectation of f with respect to the subalgebra $T^{-1}(\Sigma)$. As an operator of $L^p(\mu)$, E is the projection onto the closure of range of T and E is the identity on $L^p(\mu)$, $p \geq 1$ if and only if $T^{-1}(\Sigma) = \Sigma$. Detailed discussion of E is found in [3,4].

1.1 *paranormal

An operator $A \in B(H)$, A is said to be *paranormal if $\| A^*(x) \|^2 \leq \| A^2(x) \| \| x \|$ for all $x \in H$.

1.2 k-*paranormal

An operator $A \in B(H)$, A is said to be k-*paranormal if $\| A^*(x) \|^k \leq \| A^k(x) \| \| x \|$ for all $x \in H$.

1.3 Quasi – *paranormal

An operator $A \in B(H)$, A is said to be quasi-*paranormal if

$$\| (A^*A)(x) \|^2 \leq \| A^3(x) \| \| A(x) \|$$

for all $x \in H$ [1].

1.4 k- Quasi – *paranormal

An operator $A \in B(H)$, A is said to be k-quasi-*paranormal if

$$\| (A^*A)^k (x) \|^2 \leq \| A^{k+2}(x) \| \| A^k(x) \|$$

for all $x \in H$ [1].

1.5 (n, k) -Quasi – *paranormal

An operator $A \in B(H)$, A is said to be (n, k) -quasi-*paranormal if

$$\| A^{1+n}(A^k(x)) \|^{\frac{1}{1+n}} \| A^k(x) \|^{\frac{n}{1+n}} \geq \| A^*(A^k(x)) \|$$

for all $x \in H$ [1].

1.6 (M, k)* Class

An operator $A \in B(H)$, A is said to be $(M, k)^*$ class if $(AA^*)^k \leq A^{*k} A^k$ for $k \geq 1$.

2 Related Works in the Field

During the last thirty years several authors have defined $W_{u,T} = M_u C_T = u\,(f \circ T)$ and have studied the properties of various classes of weighted composition operators on L^2 spaces. The study of weighted composition operator was initiated.

$M_{u,T}(f) = C_T\, M_u(f) = u \circ T \;\; f \circ T$ by R. K. Singh and D. C. Kumar [2]. The concept of normality of bounded linear operators on a Hilbert Space has been generalized by different authors.

Recently, S. Senthil, P. Thangaraju and D. C. Kumar [5] have proved, the theorems on n-Normal and n-quasi-normal composite multiplication operator on L^2 -spaces. Arora and Thukral [6,7] have proved, a weighted composition operators $W_{u,T} = M_u C_T$ is *paranormal and quasi-*paranormal operators. Some results have been found by N. Chennappan and S. Karthikeyan [8], in the characterizations of *paranormal and quasi-*paranormal operators. S. Mecheri [9], has proved the results on k-quasi-paranormal operators. Many results have been found, in the characterization of k-*paranormal and (n,k) -quasi-*paranormal weighted composition operators on L^2 -spaces, see [10,11,1].

3 k -*Paranormal and $(M,k)^*$ Class Composite Multiplication Operator

Throughout the paper, by an operator we mean a bounded linear operator on a Hilbert space. If H denotes an infinite dimensional complex separable Hilbert Space, denotes the algebra of all operators on H by $B(H)$. Fahri Marevi and Muhib Lohaj [12] have proved that, for each positive integer $k \geq 2$ and define an operator A is k-*paranormal if and only if $A^{*k}A^k - k\,C^{k-1}A\,A^* + (k-1)\,C^k I \geq 0$ for all $C \geq 0$. Followed by Anuradha and Pooja Sharma [11] have characterized k-*paranormal weighted composition operators. In an analogous manner, we give a characterization of *paranormal and $(M,k)^*$ class composite multiplication operator on $L^2 -$spaces. .

3.1 Proposition

Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$. Then for $u \geq 0$

(i) $M^*{}_{u,T}\, M_{u,T}\, f = u^2 f_0\, f$

(ii) $M_{u,T}\, M^*{}_{u,T}\, f = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$

Since $\quad M_{u,T}(f) = C_T M_u(f) = u \circ T \;\; f \circ T$

$$M^n{}_{u,T}(f) = (C_T M_u)^n(f) = u_n\; (f \circ T^n)$$

the adjoint $M^*{}_{u,T}$ of $M_{u,T}$ is given by $M^*{}_{u,T}\; f = u\, f_0 \cdot E(f) \circ T^{-1}$ and

$$M^{*n}{}_{u,T} f = u f_0 \cdot E(u f_0) \circ T^{-(n-1)} \cdot E(f) \circ T^{-n}$$

where

$$E(u f_0) \circ T^{-(n-1)} = E(u f_0) \circ T^{-1} \cdot E(u f_0) \circ T^{-2} .... E(u f_0) \circ T^{-(n-1)}$$

$$E(u f_0) \circ T^{n-1} = E(u f_0) \circ T^{1} \cdot E(u f_0) \circ T^{2} .... E(u f_0) \circ T^{n-1}$$

Theorem 3.2

Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$. Then $M_{u,T}$ is *paranormal if and only if

$u^2 f_0 \cdot E(u^2 f_0) \circ T^{-1} f - 2C u^2 \circ T \cdot f_0 \circ T \cdot E(f) + C^2 I \geq 0$ almost everywhere, for all $C \geq 0$.

Proof:

Suppose $M_{u,T}$ is *paranormal. Then

$$M^{*2}_{u,T} M^2_{u,T} - 2C\, M_{u,T} M^*_{u,T} + C^2 I \geq 0 \text{ for all } C \geq 0.$$

This implies that

$$\left\langle (M^{*2}_{u,T} M^2_{u,T} - 2C M_{u,T} M^*_{u,T} + C^2 I) f, f \right\rangle \geq 0 \text{ for all } f \in L^2(\mu)$$

Since $M_{u,T}(f) = C_T M_u(f) = u \circ T \;\; f \circ T$

$$M^*_{u,T} f = u f_0 \cdot E(f) \circ T^{-1}$$

$$M^{*2}_{u,T} M^2_{u,T}(f) = u^2 f_0 \cdot E(u^2 f_0) \circ T^{-1} f$$

and we have

$$M_{u,T} M^*_{u,T} f = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$$

$$\int_E \left\{ u^2 f_0 \cdot E(u^2 f_0) \circ T^{-1} f - 2C u^2 \circ T \cdot f_0 \circ T \cdot E(f) + C^2 I \right\} d\mu \geq 0 \text{ for every } E \in \Sigma$$

$$\Leftrightarrow$$

$$u^2 f_0 \cdot E(u^2 f_0) \circ T^{-1} f - 2C u^2 \circ T \cdot f_0 \circ T \cdot E(f) + C^2 I \geq 0 \text{ almost everywhere, for all } C \geq 0$$

Corollary 3.3

If the composition operator C_T is in $B(L^2(\mu))$ then C_T is *paranormal if and only if

$f_0 \cdot E(f_0) \circ T^{-1} f - 2C\, f_0 \circ T \cdot E(f) + C^2 I \geq 0$ almost everywhere, for all $C \geq 0$.

Proof:

The proof is obtained from theorem 3.2 by putting $u = 1$.

Corollary 3.4

If the composite multiplication operator $M_{u,T}$ is in $B(L^2(\mu))$ then $M_{u,T}$ is *paranormal if and only if $u^4 \circ T \cdot f_0^2 \circ T \cdot (E(f))^2 \leq u^2 f_0 \cdot E(u^2 f_0) \circ T^{-1} f$ almost everywhere.

Proof:

Suppose $M_{u,T}$ is *paranormal is in $B(L^2(\mu))$. Then by theorem 3.2, $u^2 f_0 \cdot E(u^2 f_0)\circ T^{-1} f - 2Cu^2 \circ T \cdot f_0 \circ T \cdot E(f) + C^2 I \geq 0$ almost everywhere, for all $C \geq 0$. We know that, by elementary properties of real quadratic form, if $a > 0$, b, c are real numbers, then $at^2 + bt + c \geq 0$ for every real t if and only if $b^2 - 4ac \leq 0$.

Hence we get,

$$\Leftrightarrow (u^2 \circ T \cdot f_0 \circ T \cdot E(f))^2 \leq u^2 f_0 \cdot E(u^2 f_0)\circ T^{-1} f \text{ almost everywhere.}$$

$$\Leftrightarrow u^4 \circ T \cdot {f_0}^2 \circ T \cdot (E(f))^2 \leq u^2 f_0 \cdot E(u^2 f_0)\circ T^{-1} f \text{ almost everywhere.}$$

3.5 Example

Let $T: R \to R$ be defined by $T(x) = 1 - x$ for all $x \in R$. Then $f_0(x) = \dfrac{d\mu T^{-1}(x)}{d\mu(x)} = 1$ and $T = T^{-1}$.

Define $u: R \to R$ as $u(x) = \sqrt{\dfrac{1}{1+(x+1)^2}}$ for all $x \in R$ and $E(f) = f$.

Now, $M_{u,T}$ is *paranormal if and only if $\dfrac{3-6x}{(1+(2-x)^2)^2\ (1+(x+1)^2)} \geq 0$.

Theorem 3.6

Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$ .Then $M_{u,T}$ is k-*paranormal if and only if $u f_0 \cdot E(u f_0)\circ T^{-(k-1)} \cdot E(u_k)\circ T^{-k} f - kC^{k-1} u^2 \circ T \cdot f_0 \circ T \cdot E(f) + (k-1)C^k I \geq 0$ almost everywhere, for all $C \geq 0$.

Proof:

Suppose $M_{u,T}$ is k-*paranormal. Then

$$M^{*k}{}_{u,T} M^k{}_{u,T} - kC^{k-1} M_{u,T} M^*{}_{u,T} + (k-1)C^k I \geq 0 \text{ for all } C \geq 0.$$

This implies that

$$\left\langle (M^{*k}{}_{u,T} M^k{}_{u,T} - kC^{k-1} M_{u,T} M^*{}_{u,T} + (k-1)C^k I) f, f \right\rangle \geq 0 \quad \text{for all } f \in L^2(\mu)$$

Since $\quad M_{u,T}(f) = C_T M_u(f) = u \circ T \ \ f \circ T$

$\quad M^*{}_{u,T} f = u f_0 \cdot E(f) \circ T^{-1}$

$$M^{*k}{}_{u,T} M^{k}{}_{u,T}(f) = u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u_k) \circ T^{-k} f$$

and we have

$$M_{u,T} M^{*}{}_{u,T} f = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$$

$$\int_E \left\{ u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u_k) \circ T^{-k} f - k C^{k-1} u^2 \circ T \cdot f_0 \circ T \cdot E(f) + (k-1) C^k I \right\} d\mu \geq 0 \text{ for every } E \in \Sigma \quad .$$

$\Leftrightarrow$ $u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u_k) \circ T^{-k} f - k C^{k-1} u^2 \circ T \cdot f_0 \circ T \cdot E(f) + (k-1) C^k I \geq 0$ almost everywhere, for all $C \geq 0$.

Corollary 3.7

If the composition operator $C_T{}^k$ is in $B(L^2(\mu))$ then $C_T{}^k$ is k-*paranormal if and only if $f_0 \cdot E(f_0) \circ T^{-(k-1)} f - k C^{k-1} \cdot f_0 \circ T \cdot E(f) + (k-1) C^k I \geq 0$ almost everywhere, for all $C \geq 0$.

Proof:

The proof is obtained from theorem 3.6 by putting $u = 1$.

Fahri Marevci and Muhib Lohaj [12] have proved that, the weighted composition operator is of class $(M, k)^*$ operator. In this manner we prove the composite multiplication operator as below,

Theorem 3.8

Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$ and $k \geq 1$. Then $M_{u,T}$ is of class $(M,k)^*$ operator if and only if

$u f_0 \cdot E(u f_0) \circ T^{-(k-1)} E(u_k) \circ T^{-k} f - 2 C u^{2k} \circ T \cdot f_0{}^k \circ T \cdot E(f) + C^2\, u f_0 \cdot E(u f_0) \circ T^{-(k-1)} E(u_k) \circ T^{-k} f \geq 0$ almost everywhere, for all $C \geq 0$

Proof:

Suppose $M_{u,T}$ is of class $(M,k)^*$ operator. Then

$$M^{*k}{}_{u,T} M^{k}{}_{u,T} - 2C\,(M_{u,T} M^{*}{}_{u,T})^k + C^2 M^{*k}{}_{u,T} M^{k}{}_{u,T} \geq 0 \text{ for all } C \geq 0 .$$

This implies that

$$\left\langle (M^{*k}{}_{u,T} M^{k}{}_{u,T} - 2C\,(M_{u,T} M^{*}{}_{u,T})^k + C^2 M^{*k}{}_{u,T} M^{k}{}_{u,T}) f, f \right\rangle \geq 0 \text{ for all } f \in L^2(\mu)$$

Since $M_{u,T}(f) = C_T M_u(f) = u \circ T \;\; f \circ T$

$M^{*}{}_{u,T} f = u f_0 \cdot E(f) \circ T^{-1}$

$$M^{*k}{}_{u,T} M^{k}{}_{u,T}(f) = u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u_k) \circ T^{-k} f$$

and we have

$$M_{u,T} M^{*}{}_{u,T} f = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$$

$$\int_E \left\{ u f_0 \cdot E(u f_0) \circ T^{-(k-1)} E(u_k) \circ T^{-k} f - 2C u^{2k} \circ T \cdot f_0{}^k \circ T \cdot E(f) + C^2 u f_0 \cdot E(u f_0) \circ T^{-(k-1)} E(u_k) \circ T^{-k} f \right\} d\mu \geq 0$$

for every $E \in \sum$.

$\Leftrightarrow$

$$u f_0 \cdot E(u f_0) \circ T^{-(k-1)} E(u_k) \circ T^{-k} f - 2C u^{2k} \circ T \cdot f_0{}^k \circ T \cdot E(f) + C^2 u f_0 \cdot E(u f_0) \circ T^{-(k-1)} E(u_k) \circ T^{-k} f \geq 0$$

almost everywhere, for all $C \geq 0$.

Corollary 3.9

If the composition operator $C_T{}^k$ is in $B(L^2(\mu))$ and $k \geq 1$, then $C_T{}^k$ is of class $(M,k)^*$ operator if and only if

$f_0 \cdot E(f_0) \circ T^{-(k-1)} - 2C f_0{}^k \circ T \cdot E(f) + C^2 f_0 \cdot E(f_0) \circ T^{-(k-1)} f \geq 0$ almost everywhere, for all $C \geq 0$

Proof:

The proof is obtained from theorem 3.8 by putting $u = 1$.

Corollary 3.10

If the composite multiplication operator $M_{u,T}$ is in $B(L^2(\mu))$ and $k \geq 1$, then $M_{u,T}$ is of class $(M,k)^*$ operator if and only if $u^4 \circ T \cdot f_0{}^2 \circ T \cdot E(f) \leq u^2 f_0{}^2 \cdot (E(u f_0))^2 \circ T^{-(k-1)} \cdot (E(u_k))^2 \circ T^{-k} f$ almost everywhere.

Proof:

Suppose $M_{u,T}$ is of class $(M,k)^*$ operator on $B(L^2(\mu))$ and $k \geq 1$. Then by theorem 3.8, $u f_0 \cdot E(u f_0) \circ T^{-(k-1)} E(u_k) \circ T^{-k} f - 2C u^{2k} \circ T \cdot f_0{}^k \circ T \cdot E(f) + C^2 u f_0 \cdot E(u f_0) \circ T^{-(k-1)} E(u_k) \circ T^{-k} f \geq 0$ almost everywhere, for all $C \geq 0$.

We know that, by elementary properties of real quadratic form, if $a > 0$, b, c are real numbers, then $at^2 + bt + c \geq 0$ for every real t if and only if $b^2 - 4ac \leq 0$.

Hence we get,

$$u^4 \circ T \cdot f_0{}^2 \circ T \cdot E(f) \leq u^2 f_0{}^2 \cdot (E(u f_0))^2 \circ T^{-(k-1)} \cdot (E(u_k))^2 \circ T^{-k} f$$

almost everywhere.

Corollary 3.11

If the composition operator $C_T^{\,k}$ is in $B(L^2(\mu))$ and $k \geq 1$, then C_T is of class $(M,k)^*$ operator if and only if $f_0{}^2 \circ T \cdot E(f) \leq f_0{}^2 \cdot (E(f_0))^2 \circ T^{-(k-1)} f$ almost everywhere.

Proof:

The proof is obtained from corollary 3.10 by putting $u = 1$.

4 k-quasi-*Paranormal Composite Multiplication Operator

Ilmi Hoxha and Naim L Braha [13] have proved that, an operator A is k-quasi-*paranormal if and only if $A^{*^{k+2}} A^{k+2} - 2C A^{*^k} A A^* A^k + C^2 A^{*^k} A^k \geq 0$ for all $C \geq 0$. In an analogous manner, we derive the characterization of k-quasi-*paranormal composite multiplication operator on L^2-spaces.

Theorem 4.1

Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$. Then $M_{u,T}$ is quasi-*paranormal if and only if $u^2 f_0 \cdot E(u^2 f_0) \circ T^{-1} \cdot E(u^2 f_0) \circ T^{-2} f - 2C u^4 \circ T \cdot f_0{}^2 \circ T \cdot E(f) + C^2 u^2 f_0 f \geq 0$ almost everywhere, for all $C \geq 0$.

Proof:

Suppose $M_{u,T}$ is quasi-*paranormal. Then

$$M^{*^3}{}_{u,T} M^3{}_{u,T} - 2C\,(M_{u,T} M^*{}_{u,T})^2 + C^2 M^*{}_{u,T} M_{u,T} \geq 0 \quad \text{for all } C \geq 0.$$

This implies that

$$\left\langle (M^{*^3}{}_{u,T} M^3{}_{u,T} - 2C\,(M_{u,T} M^*{}_{u,T})^2 + C^2 M^*{}_{u,T} M_{u,T}) f, f \right\rangle \geq 0 \quad \text{for all } f \in L^2(\mu)$$

Since
$$M_{u,T}(f) = C_T M_u(f) = u \circ T \;\; f \circ T$$
$$M^*{}_{u,T} f = u f_0 \cdot E(f) \circ T^{-1}$$
$$M^{*^3}{}_{u,T} M^3{}_{u,T}(f) = u^2 f_0 \cdot E(u^2 f_0) \circ T^{-1} \cdot E(u^2 f_0) \circ T^{-2} f$$

and we have

$$M_{u,T} M^*{}_{u,T} f = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$$

$$M^*{}_{u,T} M_{u,T} f = u^2 f_0 f$$

$$\int_E \left\{ u^2 f_0 \cdot E(u^2 f_0) \circ T^{-1} \cdot E(u^2 f_0) \circ T^{-2} f - 2C u^4 \circ T \cdot f_0{}^2 \circ T \cdot E(f) + C^2 u^2 f_0 f \right\} d\mu \geq 0 \text{ for every } E \in \Sigma$$

$$\Leftrightarrow u^2 f_0 \cdot E(u^2 f_0) \circ T^{-1} \cdot E(u^2 f_0) \circ T^{-2} f - 2C u^4 \circ T \cdot f_0{}^2 \circ T \cdot E(f) + C^2 u^2 f_0 f \geq 0$$

almost everywhere, for all $C \geq 0$

Corollary 4.2

If the composite multiplication operator $M_{u,T}$ is in $B(L^2(\mu))$ then $M_{u,T}$ is quasi-*paranormal if and only if $u^8 \circ T \cdot {f_0}^4 \circ T \cdot E(f) \leq u^4 f_0 \cdot E(u^2 f_0) \circ T^{-1} \cdot E(u^2 f_0) \circ T^{-2} f$ almost everywhere.

Proof:

Suppose $M_{u,T}$ is quasi-*paranormal is in $B(L^2(\mu))$. Then by theorem 4.1,

$u^2 f_0 \cdot E(u^2 f_0) \circ T^{-1} \cdot E(u^2 f_0) \circ T^{-2} f - 2C u^4 \circ T \cdot {f_0}^2 \circ T \cdot E(f) + C^2 u^2 f_0 f \geq 0$ almost everywhere, for all $C \geq 0$

We know that, by elementary properties of real quadratic form, if $a > 0$, b, c are real numbers, then $at^2 + bt + c \geq 0$ for every real t if and only if $b^2 - 4ac \leq 0$.

Hence we get,

$$u^8 \circ T \cdot {f_0}^4 \circ T \cdot E(f) \leq u^4 f_0 \cdot E(u^2 f_0) \circ T^{-1} \cdot E(u^2 f_0) \circ T^{-2} f$$

almost everywhere.

4.3 Example

Let $T : R \to R$ be defined by $T(x) = 1 - x$ for all $x \in R$ .Then $f_0(x) = \dfrac{d\mu T^{-1}(x)}{d\mu(x)} = 1$ and $T = T^{-1}$.

Define $u : R \to R$ as $u(x) = 2x$ for all $x \in R$ and $E(f) = f$.

Now, $M_{u,T}$ is quasi-*paranormal if and only if $x^6 (1-x)^2 - (1-x)^8 \geq 0$.

Corollary 4.4

If the composition operator C_T on $B(L^2(\mu))$, then C_T is quasi-*paranormal if and only if

$${f_0}^4 \circ T \cdot E(f) \leq f_0 \cdot E(f_0) \circ T^{-1} \cdot E(f_0) \circ T^{-2} f \ .$$

Proof:

The proof is obtained from corollary 4.2 by putting $u = 1$.

Corollary 4.5

If the composition operator C_T on $B(L^2(\mu))$, then C_T is quasi-*paranormal if and only if

$$f_0 \cdot E(f_0) \circ T^{-1} \cdot E(f_0) \circ T^{-2} f - 2C {f_0}^2 \circ T \cdot E(f) + C^2 f_0 f \geq 0 \ .$$

Proof:

The proof is obtained from theorem 4.1 by putting $u = 1$.

Theorem 4.6

Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$. Then $M_{u,T}$ is k-quasi-*paranormal if and only if

$$u f_0 \cdot E(u f_0) \circ T^{-(k+1)} \cdot E(u_{k+2}) \circ T^{-(k+2)} f - 2C\, u \cdot u^2 \circ T^{-(k-1)} \cdot f_0 \cdot f_0{}^2 \circ T^{-(k-1)} \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(E(u_k)) \circ T^{-k} f + C^2 u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u_k) \circ T^{-k} f \geq 0$$

almost everywhere, for all $C \geq 0$

Proof:

Suppose $M_{u,T}$ is k-quasi-*paranormal. Then

$$M^{*k+2}{}_{u,T} M^{k+2}{}_{u,T} - 2C\; M^{*k}{}_{u,T} M_{u,T} M^{*}{}_{u,T} M^{k}{}_{u,T} + C^2 M^{*k}{}_{u,T} M^{k}{}_{u,T} \geq 0 \text{ for all } C \geq 0 .$$

This implies that

$$\left\langle (M^{*k+2}{}_{u,T} M^{k+2}{}_{u,T} - 2C\; M^{*k}{}_{u,T} M_{u,T} M^{*}{}_{u,T} M^{k}{}_{u,T} + C^2 M^{*k}{}_{u,T} M^{k}{}_{u,T})f,\ f \right\rangle \geq 0 \text{ for all } f \in L^2(\mu)$$

Since

$$M_{u,T}(f) = C_T M_u(f) = u \circ T\ \ f \circ T$$

$$M^{*}{}_{u,T} f = u f_0 \cdot E(f) \circ T^{-1}$$

$$M^{*k+2}{}_{u,T} M^{k+2}{}_{u,T}(f) = u f_0 \cdot E(u f_0) \circ T^{-(k+1)} \cdot E(u_{k+2}) \circ T^{-(k+2)} f$$

$$M^{*k}{}_{u,T} M^{k}{}_{u,T}(f) = u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u_k) \circ T^{-k} f$$

$$M^{*k}{}_{u,T} M_{u,T} M^{*}{}_{u,T} M^{k}{}_{u,T} f = u \cdot u^2 \circ T^{-(k-1)} \cdot f_0 \cdot f_0{}^2 \circ T^{-(k-1)} \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(E(u_k)) \circ T^{-k} f$$

and we have

$$M_{u,T} M^{*}{}_{u,T} f = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$$

$$M^{*}{}_{u,T} M_{u,T} f = u^2 f_0 f$$

$$\int_E \left\{ \begin{aligned} & u f_0 \cdot E(u f_0) \circ T^{-(k+1)} \cdot E(u_{k+2}) \circ T^{-(k+2)} f - 2C\, u \cdot u^2 \circ T^{-(k-1)} \cdot f_0 \cdot f_0{}^2 \circ T^{-(k-1)} \\ & \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(E(u_k)) \circ T^{-k} f + C^2 u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u_k) \circ T^{-k} f \end{aligned} \right\} d\mu \geq 0$$

for every $E \in \sum$.

$$\Leftrightarrow$$
$$u f_0 \cdot E(u f_0) \circ T^{-(k+1)} \cdot E(u_{k+2}) \circ T^{-(k+2)} f - 2C\, u \cdot u^2 \circ T^{-(k-1)} \cdot f_0 \cdot f_0{}^2 \circ T^{-(k-1)}$$
$$\cdot E(u f_0) \circ T^{-(k-1)} \cdot E(E(u_k)) \circ T^{-k} f + C^2 u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u_k) \circ T^{-k} f \geq 0$$

almost everywhere, for all $C \geq 0$

Corollary 4.7

The composition operator $C_T{}^k$ on $B(L^2(\mu))$ is k-quasi-*paranormal if and only if $f_0 \cdot E(f_0) \circ T^{-(k+1)} f - 2C\, f_0 \cdot f_0{}^2 \circ T^{-(k-1)} \cdot E(f_0) \circ T^{-(k-1)} f + C^2 f_0 \cdot E(f_0) \circ T^{-(k-1)} f \geq 0$ almost everywhere.

Proof:

The proof is obtained from theorem 4.6 by putting $u = 1$.

Corollary 4.8

If the composition operator $C_T{}^k$ on $B(L^2(\mu))$ then $C_T{}^k$ is k-quasi-*paranormal if and only if

$$f_0{}^2 \cdot f_0{}^4 \circ T^{-(k-1)} \cdot E(f_0) \circ T^{-(k-1)} f \leq f_0{}^2 \cdot E(f_0) \circ T^{-(k-1)} f$$

almost everywhere.

5 (n,k) -quasi-*Paranormal Composite Multiplication Operator

Qingping Zeng and Huaijie Zhong [1] have proved that, an operator A is (n,k) - quasi-*paranormal if and only if

$$A^{*k} A^{*1+n} A^{1+n} A^k - (1+n) C^n A^{*k} A A^* A^k + n C^{1+n} A^{*k} A^k \geq 0$$

for all $C \geq 0$. In an analogous manner, we derive the characterization (n,k) - quasi-*paranormal composite multiplication operator on L^2 -spaces.

Theorem 5.1

Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$ .Then $M_{u,T}$ is (n,k) - quasi-*paranormal if and only if

$$u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u \cdot u_k f_0) \circ T^{-k} \cdot E(u f_0) \circ T^{-(n+k)} \cdot E(u_{n+1}) \circ T^{-(n+k+1)}$$
$$-(1+n) C^n u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(E(u_k)) \circ T^{-k} \cdot u^2 \circ T^{-(k-1)} \cdot f_0 \circ T^{-(k-1)}$$
$$+ n C^{1+n} u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u_k) \circ T^{-k} \geq 0$$

almost everywhere, for all $C \geq 0$.

Proof:

Suppose $M_{u,T}$ is (n,k) -quasi-*paranormal. Then

$M_{u,T}^{*k} M_{u,T}^{*1+n} M_{u,T}^{1+n} M_{u,T}^{k} - (1+n)C^n M_{u,T}^{*k} M_{u,T} M_{u,T}^{*} M_{u,T}^{k} + nC^{1+n} M_{u,T}^{*k} M_{u,T}^{k} \geq 0$ for all $C \geq 0$.

This implies that

$$\left\langle \begin{array}{l}(M_{u,T}^{*k} M_{u,T}^{*1+n} M_{u,T}^{1+n} M_{u,T}^{k} - (1+n)C^n M_{u,T}^{*k} M_{u,T} M_{u,T}^{*} M_{u,T}^{k} \\ \qquad + nC^{1+n} M_{u,T}^{*k} M_{u,T}^{k})f,\ f\end{array} \right\rangle \geq 0$$

for all $f \in L^2(\mu)$

Since $M_{u,T}(f) = C_T M_u(f) = u \circ T \ \ f \circ T$

$$M^*_{u,T}\, f = u f_0 \cdot E(f) \circ T^{-1}$$

$$\begin{aligned} M_{u,T}^{*k} M_{u,T}^{*1+n} M_{u,T}^{1+n} M_{u,T}^{k} f &= u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u \cdot u_k f_0) \circ T^{-k} \\ &\quad \cdot E(u f_0) \circ T^{-(n+k)} \cdot E(u_{n+1}) \circ T^{-(n+k+1)} f \end{aligned}$$

$$\begin{aligned} M_{u,T}^{*k} M_{u,T} M_{u,T}^{*} M_{u,T}^{k} f &= u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(E(u_k)) \circ T^{-k} \\ &\quad \cdot u^2 \circ T^{-(k-1)} \cdot f_0 \circ T^{-(k-1)} f \end{aligned}$$

and we have

$$M^{*k}_{u,T} M^k_{u,T}(f) = u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u_k) \circ T^{-k} f$$

$$M_{u,T} M^*_{u,T}\, f = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$$

$$M^*_{u,T} M_{u,T} f = u^2 f_0 f$$

$$\int_E \left\{ \begin{array}{l} u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u \cdot u_k f_0) \circ T^{-k} \cdot E(u f_0) \circ T^{-(n+k)} \cdot E(u_{n+1}) \circ T^{-(n+k+1)} \\ -(1+n)C^n u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(E(u_k)) \circ T^{-k} \cdot u^2 \circ T^{-(k-1)} \cdot f_0 \circ T^{-(k-1)} \\ + nC^{1+n} u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u_k) \circ T^{-k} \end{array} \right\} d\mu \geq 0$$

for every $E \in \sum$.

$$\Leftrightarrow$$

$$\begin{aligned} & u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u \cdot u_k f_0) \circ T^{-k} \cdot E(u f_0) \circ T^{-(n+k)} \cdot E(u_{n+1}) \circ T^{-(n+k+1)} \\ & -(1+n)C^n u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(E(u_k)) \circ T^{-k} \cdot u^2 \circ T^{-(k-1)} \cdot f_0 \circ T^{-(k-1)} \\ & + nC^{1+n} u f_0 \cdot E(u f_0) \circ T^{-(k-1)} \cdot E(u_k) \circ T^{-k} \geq 0 \end{aligned}$$

almost everywhere, for all $C \geq 0$

Corollary 5.2

If the composition operator $C_T^{(n,k)}$ on $B(L^2(\mu))$, then $C_T^{(n,k)}$ is (n,k) -quasi-*paranormal if and only if

$$f_0 \cdot E(f_0) \circ T^{-(k-1)} \cdot E(f_0) \circ T^{-k} \cdot E(f_0) \circ T^{-(n+k)} - (1+n)C^n f_0 \cdot E(f_0) \circ T^{-(k-1)} \cdot f_0 \circ T^{-(k-1)}$$
$$+ nC^{1+n} f_0 \cdot E(f_0) \circ T^{-(k-1)} \geq 0$$

almost everywhere, for all $C \geq 0$.

Proof:

The proof is obtained from theorem 5.1 by putting $u = 1$.

Competing Interests

Authors have declared that no competing interests exist.

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