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Development of a Lyapunov Exponent Based Chaos Diagram in the Parameter Plane of Logistic Map

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Authors' contributions

Author TAOS conceived the study plan, developed and validated the computing codes used. Thereafter he used the validated codes to simulate the findings of this study and wrote the first manuscript draft except the introduction. Author OOA convinced by the study plan wrote the introduction and put the agreed first manuscript draft in format provided by the Journal. The two authors jointly implemented all the suggestions made by the reviewers.

Original Research Article

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ABSTRACT

A one dimensional model of population growth called logistic map can be used as platform for introducing beginners to the phenomenon of chaos and nonlinear dynamics. Despite the simplicity of logistic map, it has been used with success for the introduction of fixed point attractor, periodic and aperiodic responses, sensitivity to initial conditions, return map and bifurcation diagram. This understanding motivated the present study to develop two dimensional chaos diagram for a one dimensional logistic map as a way of introducing the beginners to the theories of fractals and chaos. Simulation of unsteady solutions, steady solutions and its corresponding Lyapunov exponent characterisation of logistic map were effected for selection of drive parameters for various grid resolutions, constant step size and at one grid point a time from 0.3 and 0.5 initial conditions. Validation were made of graphical results of parameter versus Lyapunov exponent and the parabola-attractors. The Lyapunov exponent characterisation results were grouped into three classifications: divergence, periodic and chaotic based on average Lyapunov value.

There is qualitative agreement of validation results. The total number of grid points with divergence or periodic or chaotic response increases exponentially with increasing resolutions. The zoomed parameters counterpart has average 0%, 60% and 40% of

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divergent, periodic and chaotic results respectively across resolutions. The corresponding chaos diagram exhibited fractal structures by its layers of order within chaos as can be found in the bifurcation diagrams of nonlinear dynamical systems.

Keywords: Lyapunov exponent; Chaos diagram; Logistic map; fractal structures; Drive parameters.

1. INTRODUCTION

Logistic map has been widely reported in the literature as a nonlinear difference equation that has strong manifestation for chaos. The applications of logistic map in engineering and technology is enormous. The focus of paper [1] is on design of a chaotic noise generator governed by a logistic map. The performance evaluation results demonstrate correct operation of the analog noise generating circuit system. Paper [2] examined practical applications of chaos theory in microelectronic field using logistic map. It was found in their study that logistic map modelling is highly promising when genetic algorithm is used for numerical simulations. Moreover, paper [3] has shown that some nonlinear systems that have their subdomains on the logistic map are often characterised with unique chaotic attractors. These chaotic attractors have interesting stabilising characteristics. This indeed is a good contribution to the literature on the exciting dynamic of logistic model.

Recent research proposes that, the use of Lyapunov exponents for nonlinear dynamics characterisation have attracted special interests among research scholars. In paper [4], Lyapunov exponent was employed in the study of chaos of Duffing's equation that is perturbed by white noise. The results of the study showed that the stability and chaotic dynamics of a nonlinear system is strongly influenced by the Lyapunov exponents. Papers [5-8] have further shown the importance of Lyapunov exponent as a characterisation tool in nonlinear system dynamics.

Extensive search in the open access literature shows that only few works have utilised Lyapunov exponent based chaos diagram in their studies. Papers [9-11] which focus on Lyapunov exponent based chaos diagram have served as satisfactory platform for the present paper. Paper [9] reported a two-dimensional bifurcation diagram using largest Lyapunov exponent codified in a continous range of colours for a driven chaotic oscillator with complex variable. The periodic, quasiperiodic and chaotic behaviours depicted in the colourful diagram further attests to the fact that chaos based diagram is resourceful for enhancing results interpretations quality. Lyapunov exponent based chaos diagram was equally adopted in paper [10] study. The authors satisfactorily developed a Lyapunov exponent based graph of two-parameters map. In the bistability region of the Lyapunov exponent based graph, there is a strong manifestation of sensitivity to initial conditions which is one of the key characteristics of chaos.

In the article posted by paper [11], some efforts have been made at laboratory level on the utility of Lyapunov exponent based-chaos diagram for Logistic map dynamics study. Obviously, there is a siginificant research gap in the application of chaos diagram for Logistic map dynamics characterisation. The present paper which is strongly motivated by the quest to introduce the beginners to the theories of chaos and nonlinear dynamics focus on the development of Lyapunov exponent based-chaos diagram in one dimensional logistic map.

2. METHODOLOGY

Paper [12] provided a discrete model of limited growth (logistic map) relating respectively the rescaled growth measure at current and previous time (x_{n+1} and x_n) as in equation (1) for value of r between 1 and 4.

The term $(1-x_n)$ serves as growth inhibitor because as x_n approaches 1, this term approaches zero.

$$x_{n+1} = rx_n(1-x_n) \tag{1}$$

The motivation for the present study is in part derived from the expansion of equation (1) as suggested by paper [12] to produce first equation (2) and then equation (3) upon replacement of the two r's with separate parameters, a and b.

$$x_{n+1} = rx_n - rx_n^{\ 2} \tag{2}$$

$$x_{n+1} = ax_n - bx_n^{\ 2} \tag{3}$$

Further motivation for this study come from paper [13]. Therefore, the present study focus on FORTRAN90 simulation and Lyapunov exponent characterisation of the dynamics of logistic model using equation (3) for selected grid points from the parameter plane defined by $1 \le a, b \le 4$. According to paper [14], an explicit average Lyapunov exponent (λ) is defined as in equation (4). For the present study, the appropriate $f(x_n)$ is related to x_{n+1} as in equation (5). Therefore the rate $\frac{df(x_n)}{dx_n}$ require in equation (4) can be obtained by evaluating the function equation (6) using $f(x_n)$ from equation (5).

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{n=N} \log_2 \left| \frac{df(x_n)}{dx_n} \right| \tag{4}$$

$$x_{n+1} = ax_n - bx_n^{\ 2} = f(x_n) \tag{5}$$

$$\frac{df(x_n)}{dx_n} = a - 2bx_n \tag{6}$$

FORTRAN90 coded programs used for this study was validated in two parts using reported graphical result of Lyapunov exponent versus control parameter ($3 \le a \le 4$) by paper [14] and the parabola attractor by paper [15]. Although there is no significant variation in the procedures for the estimation of Lyapunov exponent for the old and new methods (equation 1: single parameter driven and equation 3: double parameter driven) however the parameters selection space for the new method enjoyed exponential increase relative to its old method counterpart. Thus more experimental points are feasible in the new method compared to the old method.

2.1 Simulation Parameters

The numbers of unsteady and steady solutions are 100 and 2000 respectively from two initial conditions 0.3 and 0.5. The grid resolution is between 11×11 and 201×201 at constant step of 10. A grid is characterised as diverging when the solution returned is equal or greater than the set 10-unit tolerance limit. That is; divergent situation implies $X_{n+1} \geq 10$ after few iteration time steps. Similarly, a grid is marked periodic when the returned average Lyapunov value is equal or lesser than zero and chaotic when this is greater than zero. In addition, the returned value of X_{n+1} with increasing iteration time step for both periodic and chaotic are bounded ($0 \leq X_{n+1} \leq 1$).The chaos diagram in the present study is the scatter plot of the grid points with average Lyapunov exponent greater than zero.

3. RESULTS AND DISCUSSION

The validation results obtained at 0.3 and 0.5 initial conditions are insignificantly different. They are qualitatively the same as in respective Figs. 1 and 2 and compare qualitatively well with the literature reference standard paper [14]. Similarly, Fig. 3 compare qualitatively well with the parabola-attractor in the x_n , x_{n+1} -plane by paper [15].

With reference to Table 1 , the total grid points obtained for chaotic, periodic and divergent response increases with increase in the number of grid points per parameter axis (a or b) and for the two initial conditions studied. The sum total of grids points (chaotic, periodic and divergent) add up to the square of number of grid points per parameter axis for all resolutions as expected. Furthermore, the observed increase in the total grid points per each response classification follow a power law as in Fig. 3. However, the physical spread of these points on the parameter plane $1 \leq a, b \leq 4$ for the chaotic response is termed chaotic diagram as provided in Fig. 4. The chaos diagrams are qualitatively the same but qualitative and quantitative differences are apparent by visual evaluation especially in the parameter range $.1.0 \leq a \leq 3.5$. Therefore, it can be concluded that chaos diagram of the logistic map has noticeable sensitivity to initial conditions.

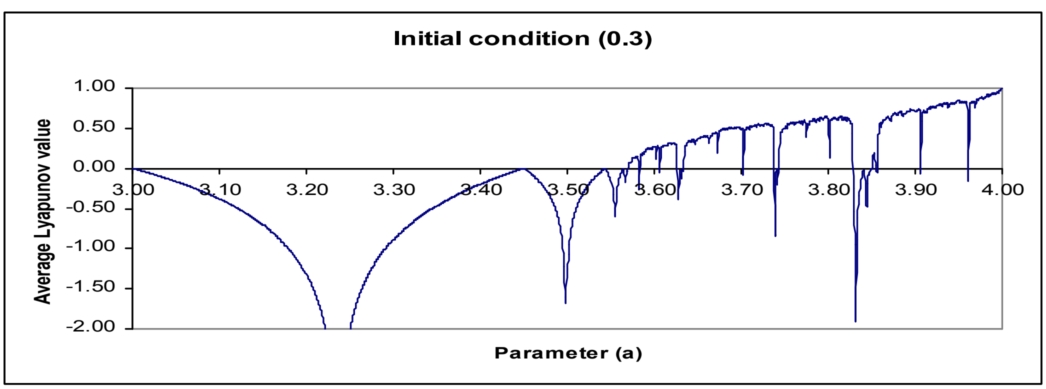

Fig. 1. Average Lyapunov value versus control parameter (a) for the logistic map at 0.3 initial condition

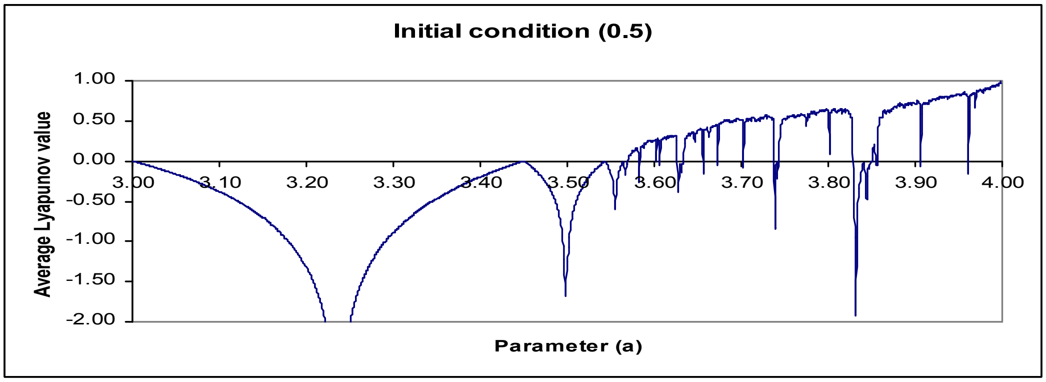

Fig. 2. Average Lyapunov value versus control parameter (a) for the logistic map at 0.5 initial condition

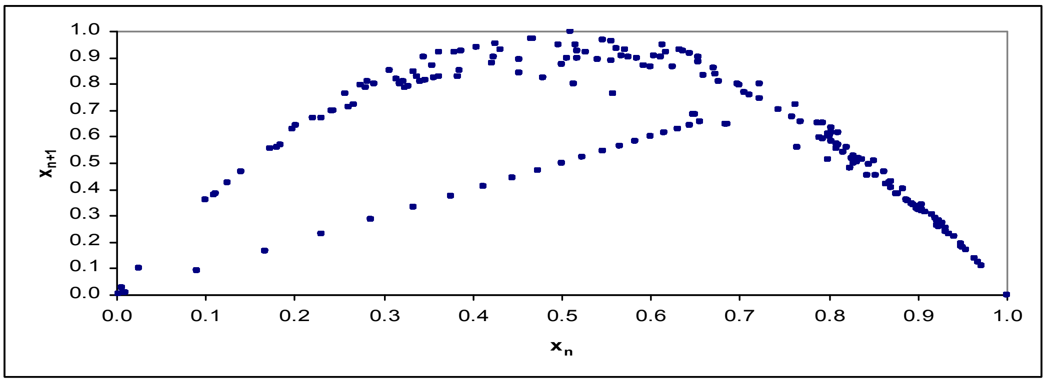

Fig. 3. Trace of the parabola-attractor in the the x_n, x_{n+1}-plane for the logistic map with $1 \leq a, b \leq 4$ at grid resolution of 31×31 starting from 0.3 initial condition

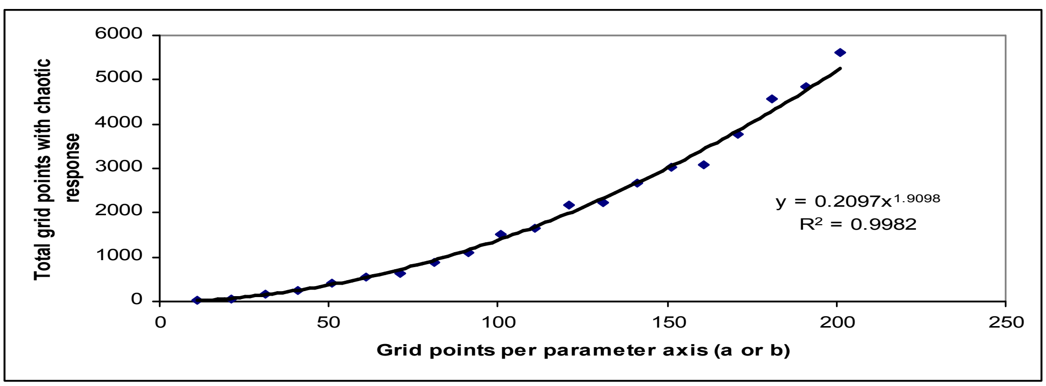

Fig. 4. Total grid points with chaotic response versus grid points per parameter axis for 0.3 initial condition simulation

Table 1. Results classification variation details for parameter plane defined by $1 \le a, b \le 4$

Grid points per axis (a or b)	Response at parameters grid points and total by classification					
	Initial condition (0.3)			Initial condition (0.5)		
	Chaotic	Periodic	Divergent	Chaotic	Periodic	Divergent
11	22	96	3	22	83	16
21	63	371	7	63	322	56
31	156	794	11	170	679	112
41	246	1417	18	246	1239	196
51	408	2166	27	408	1887	306
61	552	3134	35	578	2719	424
71	639	4355	47	639	3826	576
81	891	5610	60	892	4913	756
91	1106	7103	72	1135	6214	932
101	1515	8597	89	1515	7530	1156
111	1665	10549	107	1665	9250	1406
121	2183	12335	123	2231	10762	1648
131	2227	14790	144	2227	12998	1936
141	2679	17035	167	2679	14946	2256
151	3024	19590	187	3087	17157	2557
161	3082	22626	213	3078	19927	2916
171	3764	25237	240	3762	22173	3306
181	4558	27939	264	4610	24487	3664
191	4852	31334	295	4836	27549	4096
201	5628	34446	327	5628	30217	4556

Tables 2 and 3 illustrated variation of Lyapunov exponent for different combination of drive parameters (a, b) in logistic map. From these tables it can be observed that positive and negative Lyapunov exponent are respectively indication of chaotic and periodic response of the logistic map. However for divergent behavior the Lyapunov exponent is not computable. Figs. 1, 2 and 5 collectively suggested that interesting chaos diagrams existed in the parameter range $3 \le a, b \le 4$ for either 0.3 or 0.5 initial condition. These are demonstrated in the next Figs. 6 to 9. By visual comparison, Figs. 7 and 9 are qualitatively the same and each manisfested windows for order within chaos as illuminated by the alternate strip layers of chaotic and periodicity.

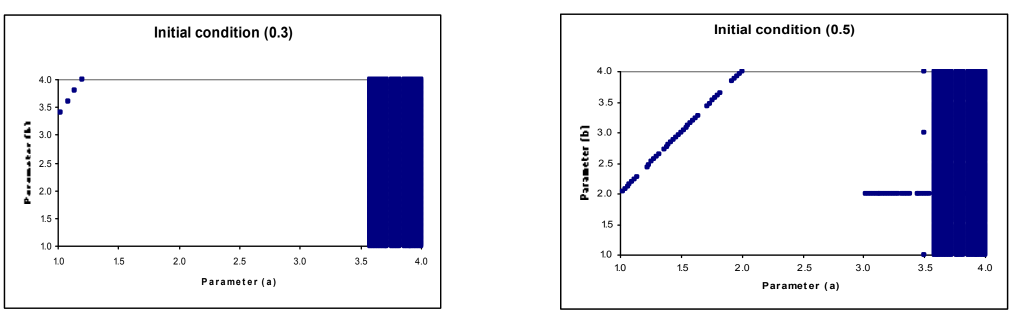

Fig. 5. Chaos diagrams of logistic map for parameter range $1 \le a, b \le 4$ at 151×151 resolution

Table 2. Selected responses details for initial condition of 0.3 at 151×151 resolution

Response by Classification								
Parameters/Average Lyapunov value (λ)								
Chaotic			Periodic			Divergent		
a	b	λ	a	B	λ	a	b	λ
1.020	3.400	0.029	1.000	1.000	-0.004	1.000	3.340	NC
1.080	3.600	0.111	1.000	1.020	-0.004	1.000	3.360	NC
1.140	3.800	0.189	1.000	1.040	-0.004	1.000	3.380	NC
1.200	4.000	0.263	1.000	1.060	-0.004	1.000	3.400	NC
3.580	1.000	0.154	1.000	1.080	-0.004	1.000	3.420	NC
3.580	1.020	0.153	1.000	1.100	-0.004	1.000	3.440	NC
3.580	1.040	0.154	1.000	1.120	-0.004	1.000	3.460	NC
3.580	1.060	0.153	1.000	1.140	-0.004	1.000	3.480	NC
3.580	1.080	0.158	1.000	1.160	-0.004	1.000	3.500	NC
3.580	1.100	0.158	1.000	1.180	-0.004	1.000	3.520	NC

Note: That NC is the same as not computable.

Table 3. Selected responses details for initial condition of 0.5 at 151×151 resolution

Response by classification								
Parameters/Average Lyapunov value (λ)								
Chaotic			Periodic			Divergent		
a	b	λ	a	B	λ	a	b	λ
1.020	2.040	0.029	1.000	1.000	-0.004	1.000	2.020	NC
1.040	2.080	0.057	1.000	1.020	-0.004	1.000	2.040	NC
1.060	2.120	0.084	1.000	1.040	-0.004	1.000	2.060	NC
1.080	2.160	0.111	1.000	1.060	-0.004	1.000	2.080	NC
1.100	2.200	0.138	1.000	1.080	-0.004	1.000	2.100	NC
1.120	2.240	0.164	1.000	1.100	-0.004	1.000	2.120	NC
1.140	2.280	0.189	1.000	1.120	-0.004	1.000	2.140	NC
1.220	2.440	0.287	1.000	1.140	-0.004	1.000	2.160	NC
1.240	2.480	0.310	1.000	1.160	-0.004	1.000	2.180	NC
1.260	2.520	0.333	1.000	1.180	-0.004	1.000	2.200	NC

Note: That NC is the same as not computable.

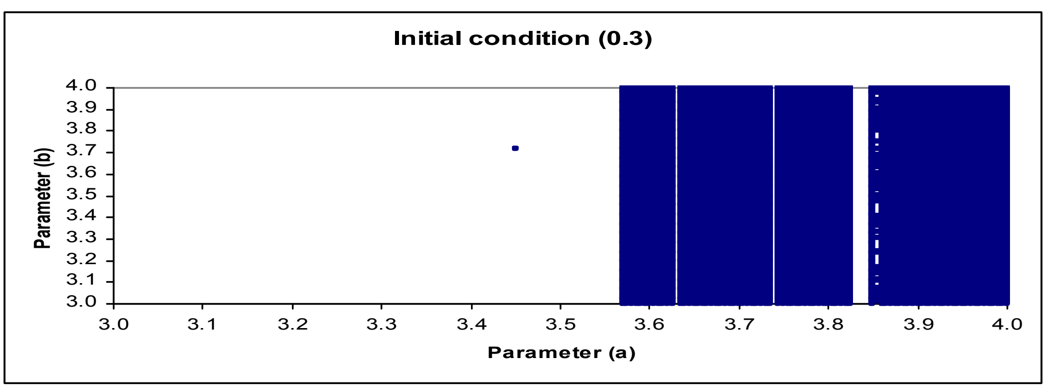

Fig. 6. Chaos diagrams of logistic map for parameter range $3 \leq a,b \leq 4$ at 201×201 resolution from 0.3 initial condition

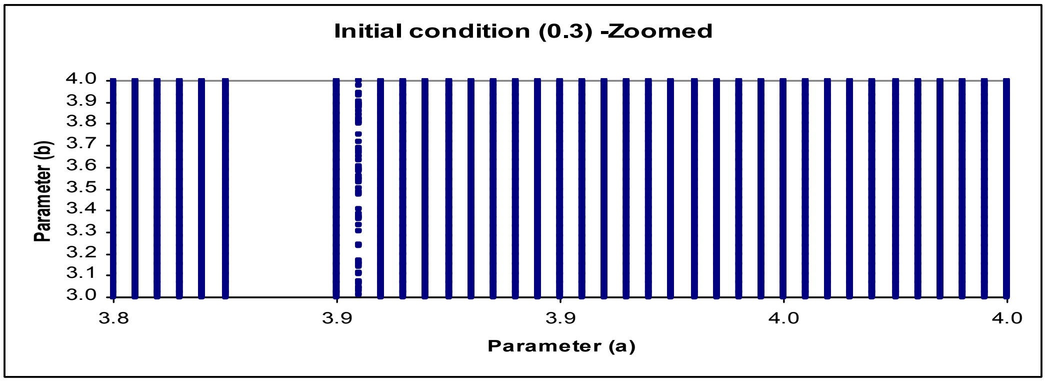

Fig. 7. Chaos diagrams of logistic map for parameter range $3.8 \le a \le 4$ and $3 \le b \le 4$ and at 201×201 resolution from 0.3 initial condition

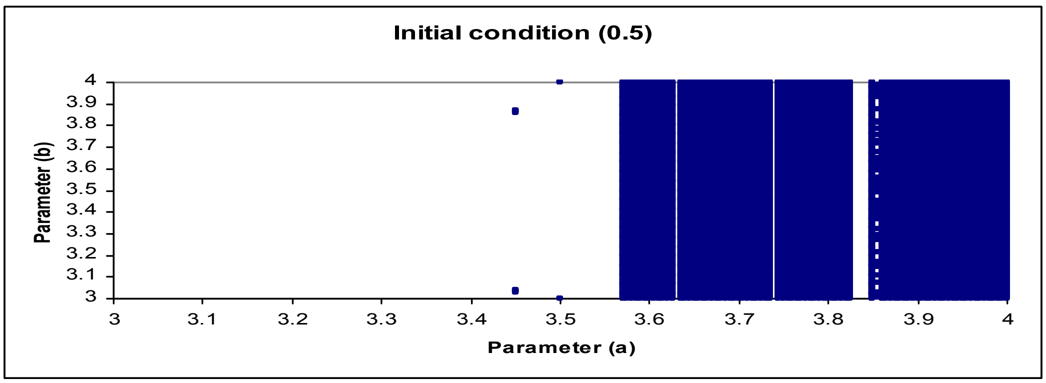

Fig. 8. Chaos diagrams of logistic map for parameter range $3 \le a, b \le 4$ at 201×201 resolution from 0.5 initial condition

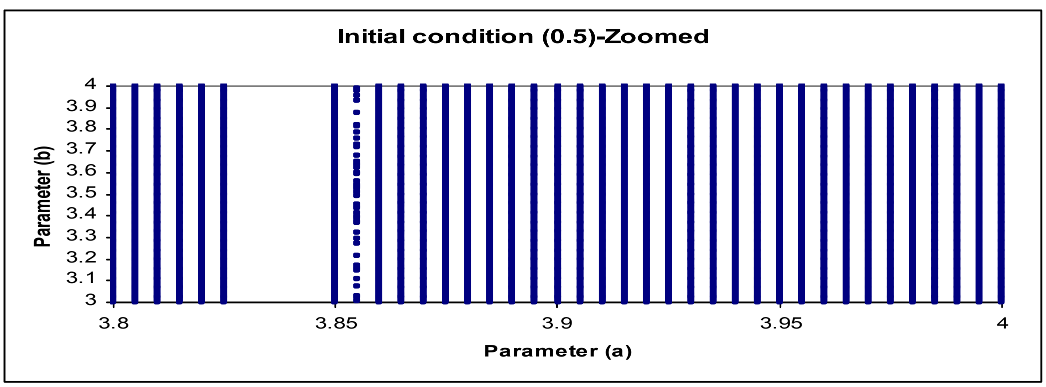

Fig. 9. Chaos diagrams of logistic map for parameter range $3.8 \le a \le 4$ and $3 \le b \le 4$ and at 201×201 resolution from 0.5 initial condition

Table 4. Percentage variation of responses classification with increasing resolution for 0.3 initial condition and $3 \le a, b \le 4$

Grid points per axis (a or b)	Response at parameters grid points and total/percentage by classification					
	Total			Percentage		
	Chaotic	Periodic	Divergent	Chaotic	Periodic	Divergent
11	55	66	0	45.5	54.5	0.0
21	189	252	0	42.9	57.1	0.0
31	384	577	0	40.0	60.0	0.0
41	738	943	0	43.9	56.1	0.0
51	1020	1581	0	39.2	60.8	0.0
61	1552	2169	0	41.7	58.3	0.0
71	1917	3124	0	38.0	62.0	0.0
81	2675	3886	0	40.8	59.2	0.0
91	3297	4984	0	39.8	60.2	0.0
101	4040	6161	0	39.6	60.4	0.0
111	4884	7437	0	39.6	60.4	0.0
121	5984	8657	0	40.9	59.1	0.0
131	6568	10593	0	38.3	61.7	0.0
141	7898	11983	0	39.7	60.3	0.0
151	9087	13714	0	39.9	60.1	0.0
161	10010	15911	0	38.6	61.4	0.0
171	10879	18362	0	37.2	62.8	0.0
181	12748	20013	0	38.9	61.1	0.0
191	14188	22293	0	38.9	61.1	0.0
201	16137	24264	0	39.9	60.1	0.0

Table 5. Percentage variation of responses classification with increasing resolution for 0.5 initial condition and $3 \le a, b \le 4$

Grid points per axis (a or b)	Response at parameters grid points and total/percentage by classification					
	Total			Percentage		
	Chaotic	Periodic	Divergent	Chaotic	Periodic	Divergent
11	57	64	0	47.1	52.9	0.0
21	191	250	0	43.3	56.7	0.0
31	374	587	0	38.9	61.1	0.0
41	740	941	0	44.0	56.0	0.0
51	1022	1579	0	39.3	60.7	0.0
61	1529	2192	0	41.1	58.9	0.0
71	1920	3121	0	38.1	61.9	0.0
81	2677	3884	0	40.8	59.2	0.0
91	3260	5021	0	39.4	60.6	0.0
101	4046	6155	0	39.7	60.3	0.0
111	4891	7430	0	39.7	60.3	0.0
121	5933	8708	0	40.5	59.5	0.0
131	6557	10604	0	38.2	61.8	0.0
141	7901	11980	0	39.7	60.3	0.0
151	9025	13776	0	39.6	60.4	0.0
161	10002	15919	0	38.6	61.4	0.0
171	10877	18364	0	37.2	62.8	0.0
181	12656	20105	0	38.6	61.4	0.0
191	14200	22281	0	38.9	61.1	0.0
201	16134	24267	0	39.9	60.1	0.0

As shown in Tables 4 and 5 , there is complete nonavailability of divergent response across all studied resolutions and within the parameter range of $3 \leq a, b \leq 4$. The average across all resolutions of the percentage ratio of chaotic to period responses is approximately 40.0 to 60.0. Thus this parameter plane is periodically dominated as can be seen in Figs. 7 and 9.

4. CONCLUSION

This study has shown that the simple one dimensional logistic map can be utilised to launch interested beginners to the concept of chaos diagram, a basic concept in nonlinear dynamics and chaos. Its parameter $3 \leq a, b \leq 4$ plane investigated by Lyapunov exponent consist respectively an average percentage ratio of 40.0 to 60.0 grid points with chaotic and periodic responses while there is no grid point that exhibited divergence. Interestingly, the chaos diagram developed exhibited fractal structures by its layers of order within chaos as can be found in the bifurcation diagrams of nonlinear dynamical systems. One of the potential applications of the present study is that it can be adapted to simulate the response of an hypertensive patient (a nonlinear system) when he/she is to be placed on different dose combinations of two medically certified hypertensive drugs (labelled a & b). The dose spectrum for the drugs can be likened to the ranges of parameters a and b (i.e. $1 \leq a, b \leq 4$). The patient three expected responses namely: Hypertension controlled; Hypertension indifferent/neutral; and Hypertension worsen can be likened to chaotic, periodic and divergent reported in the present study.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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