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On Propositions Pertaining to the Riemann Hypothesis

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Short Research Article

Abstract

In this paper, we establish some methods and propositions that allows a study of the Riemann Hypothesis. The main idea is to divide a sum (finite or infinite), so that the two parts are not equivalent in some sense and hence lead to a non-zero point. Particularly so, we suggest methods to establish how different sub-regions (subsets) may correspond to non-zeroes.

Keywords: Riemann hypothesis; riemann's zeta-function; prime number.

2010 Mathematics Subject Classification: 30A99, 30E99

1 Introduction

In this paper, we study the Riemann zeta function ([1], [2], [3]) from the point of view of establishing a method to determine whether a point would qualify as a non-zero point. This is based on prior perspectives as presented in [4]. The results perhaps might be viewed as complementary to theorems in [5], [6] and [7]. As noted in [4], the previous papers and theorems related to the Riemann Hypothesis may be found in [8],[9],[10],[11] and [12].

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2 Propositions and the Expectation Representation

The Riemann zeta function is defined as follows in this paper. Let us first define the region of interest explicitly as a subset of $\mathbb{R}^2$ as

$$S = \{(\sigma, t) \in \mathbb{R}^2 : \sigma \in (0, 1); t \neq 0\}. \tag{2.1}$$

The riemann zeta function is defined for each $s \in S$ as

$$\zeta(s) := \sum_{n=1}^{\infty}(1 - \frac{1}{2^{1-s}}) \times \frac{(-1)^{n+1}}{n^s}, \tag{2.2}$$

where in the infinite sum, we have the vector

$$\frac{1}{n^s} = e^{-\sigma \ln(n)}(\cos(-t \ln(n)), \sin(-t \ln(n))), \tag{2.3}$$

obtained using Euler's formula [3],[13], multiplied by the vector $1 - \frac{1}{2^{1-s}}$ as defined by the binary operation of multiplication of complex numbers i.e. $\mathbb{R}^2$. This is defined for points $s = (\sigma, t)$ and $s' = (\sigma', t')$ in $\mathbb{R}^2$ as $s + s' = (\sigma + \sigma', t + t')$ and $s \times s' := (\sigma\sigma' - tt', \sigma t' + t\sigma')$. Since for each $s \in S$, we have that $1 - \frac{1}{2^{1-s}} \neq 0$, it follows that the zeroes of ζ coincide exactly with the zeroes of the function ζ^* given only by the alternating Dirichlet sum [3],[13] i.e.

$$\zeta^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}. \tag{2.4}$$

We now make a digression and prove a preliminary proposition that will be needed.

Proposition 2.1. *For each $\sigma \in (0, 1)$,*

$$\sup_{x \in (0,1)} \frac{x}{e^{\sigma x} - 1} = \frac{1}{\sigma}. \tag{2.5}$$

Proof. We note that by applying the fundamental theorem of calculus, for any differentiable f, g on $[0, 1)$ such that $g(x) \neq 0$ for each x and $f(0) = g(0) = 0$, we have that for each $x > 0$,

$$\frac{f(x)}{g(x)} = \frac{\int_{(0,x)} f'(y)dy}{\int_{(0,x)} g'(y)dy}. \tag{2.6}$$

Now we define $f(x) = x$ and $g(x) = e^{\sigma x} - 1$. Since, $f'(x) \leq \frac{g'(x)}{\sigma}$, from 2.6, we have that

$$\frac{f(x)}{g(x)} \leq \frac{1}{\sigma}. \tag{2.7}$$

Further, by applying L'Hospital's rule, it follows that $\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{1}{\sigma}$. Hence, we obtain 2.5. □

We now study the defined function ζ^*. Suppose we define the vector Z_0 and for each $s \in S$, we define the sequence $\{Z(s)_n\}_{n\in\mathbb{Z}^+}$ as follows

$$Z_0 = (1, 0); \tag{2.8}$$

$$Z_n(s) = \frac{1}{(2n + 1)^s} - \frac{1}{(2n)^s}; \text{ for each } n \in \mathbb{Z}^+. \tag{2.9}$$

Hence, the function ζ^* may expressed as

$$\zeta^*(s) = Z_0 + \sum_{n=1}^{\infty} Z_n(s). \tag{2.10}$$

We may now prove the following proposition.

Proposition 2.2. *The series in 2.10 converges absolutely.*

Proof. Let $s = (\sigma, t) \in S$. Given the nature and symmetry of the cos and sine functions, we may assume without loss of generality that $t > 0$. Let $n_0 \in \mathbb{Z}^+$ such that for each $n \geq n_0$, we have that $(\ln(2n+1) - \ln(2n))t \leq \frac{\pi}{2}$. This means that beyond the point n_0, the angle in radians traversed in $\mathbb{R}^2$ between the vectors $\frac{1}{(2n+1)^s}$ and $\frac{1}{(2n)^s}$ which is the value $(\ln(2n+1) - \ln(2n))t$ is at most $\frac{\pi}{2}$ and is hence, an acute angle.

Now, define the angle

$$\theta_n = (\ln(2n+1) - \ln(2n))t. \tag{2.11}$$

Geometrically, we may prove using trignometric relations, that

$$||Z_n(s)|| = ||\frac{1}{(2n+1)^s} - \frac{1}{(2n)^s}|| \tag{2.12}$$

$$= \sqrt{\Big(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\Big)^2 + \frac{2}{(2n+1)^\sigma(2n)^\sigma}(1 - \cos(\theta_n))}. \tag{2.13}$$

Again, geometrically we may prove that distance $||\frac{1}{(2n+1)^s} - \frac{1}{(2n)^s}||$ is at most the difference $\Big(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\Big)$ plus the arc length $\theta_n \frac{1}{(2n+1)^\sigma}$, which corresponds to the circle centered at zero and has radius $\frac{1}{(2n+1)^\sigma}$. This is obtained using the triangular inequality. The arc length is greater than the distance between the vector $\frac{1}{(2n+1)^s}$ and the unique point z on the line segment conv$\{0, \frac{1}{(2n)^s}\}$ such that $||z|| = \frac{1}{(2n+1)^\sigma}$. Note that $||z - \frac{1}{(2n)^s}|| = \Big(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\Big)$. Then, we apply the triangular inequality with the vectors $\frac{1}{(2n+1)^s}$, $\frac{1}{(2n)^s}$ and z. Hence, we obtain that for each $n \geq n_0$,

$$||Z_n(s)|| \leq \Big(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\Big) + \theta_n \frac{1}{(2n+1)^\sigma}. \tag{2.14}$$

We now apply proposition 2.1, by setting $x = (\ln(2n+1) - \ln(2n))$, we obtain the inequality

$$\theta_n \frac{1}{(2n+1)^\sigma} \leq \frac{t}{\sigma}\Big(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\Big) \tag{2.15}$$

Hence, for each $n \geq n_0$, we have that

$$||Z_n(s)|| \leq \Big(1 + \frac{t}{\sigma}\Big)\Big(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\Big). \tag{2.16}$$

Since the series $\sum_{n\geq 1}\Big(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\Big)$ converges absolutely, it follows from 2.16 that the series in 2.10 converges absolutely. □

Based on the proof of the above proposition, we define the function

$$F(\sigma) := \sum_{n\geq 1}\Big(\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\Big). \tag{2.17}$$

We prove the following proposition.

Proposition 2.3. *Suppose that $s = (\sigma, t) \in S$ such that $t \in [-\frac{\pi}{2\ln(3/2)}, \frac{\pi}{2\ln(3/2)}]$. If*

$$\Big(1 + \frac{|t|}{\sigma}\Big)F(\sigma) < 1, \tag{2.18}$$

then $\zeta(s) \neq 0$.

Proof. The proof follows from the proof of proposition 2.2. Assume, without any loss of generality that $t > 0$. Since $t \leq \frac{\pi}{2\ln(3/2)}$, this means that for each $n \geq 1$, the angle θ_n is acute. Hence, the upper bound in 2.16 applies. Hence, we have that

$$\sum_{n\geq 1} ||Z_n(s)|| \leq \Big(1 + \frac{t}{\sigma}\Big)F(\sigma) < 1. \tag{2.19}$$

Suppose for contradiction $\zeta(s) = 0$. Then, we get that $||Z_0|| = ||\sum_{n\geq 1} Z_n(s)|| = 1$. Hence, we get from 2.19 that $1 > \sum_{n\geq 1} ||Z_n(s)|| \geq ||\sum_{n\geq 1} Z_n(s)|| = 1$, which is a contradiction. □

This implies the following proposition.

Proposition 2.4. *If $s = (\sigma, t) \in [\frac{1}{2}, 1) \times \Big(\frac{1-\sqrt{2}}{2}, \frac{\sqrt{2}-1}{2}\Big)$, then $\zeta(s) \neq 0$.*

Proof. Follows from proposition 2.3 since $F(\sigma) \leq \frac{1}{2^\sigma}$. □

The expectations representation: In [4], it was shown that identifying zeroes of the riemann zeta function may be interpretated as a problem of identifying a zero expectation random vector. This may be expressed as follows. If a series $\sum_{n\geq 1} z_n$ is absolutely convergent, then we define on the set of positive integers $\mathbb{Z}^+$, a probability measure $\mu(\{n\}) := \frac{||z_n||}{\sum_{m\geq 1}||z_m||}$ and random vector $X(n) := z_n/||z_n||$. Then, we have that the expectation $\mathbb{E}_\mu[X] = 0$ if and only if $\sum_{n\geq 1} z_n = 0$. In this paper, we are interested in the sequence pertaining to 2.9 i.e $z_1 := Z_0$ and $z_n := Z_{n-1}(s)$ for each $n \geq 2$. The vector $Z_n(s)$ may be conveniently represented via geometry in polar form (see [3],[13]) when θ_n is acute, as for the defined angle

$$\hat{\theta}_n := \tan^{-1}\Big(\frac{\frac{1}{(2n+1)^\sigma}\sin(\theta_n)}{\frac{1}{(2n)^\sigma} - \frac{1}{(2n+1)^\sigma}\cos(\theta_n)}\Big), \tag{2.20}$$

we have that

$$Z_n(s) = ||Z_n(s)||(\cos(-\ln(n)t + \hat{\theta}_n), \sin(-\ln(n)t + \hat{\theta}_n)). \tag{2.21}$$

Proposition 2.3 in this paper and Proposition 2.4 from [4] hence can be obtained by an application of a more general proposition, which we next prove. For any $z \in \mathbb{R}^2$, denote as $\theta(z) \in [0, 2\pi]$, the angle in radians, traversed by the vector z in its polar form.

Proposition 2.5. *Let μ be a probability measure on the unit circle $\mathbb{S}^1$. Then,*

$$\mathbb{E}_\mu[z] \neq 0 \text{ if there exist numbers } 0 \leq \theta' \leq \theta'' \leq 2\pi \text{ such that } \theta' - \theta \leq \frac{\pi}{2} \text{ and}$$
$$\mu(\{z : \theta' \leq \theta(z) \leq \theta''\}) > \frac{1}{1 + \cos\Big(\frac{\theta''-\theta'}{2}\Big)}. \tag{2.22}$$

Proof. Suppose that 2.22 holds. Then, by geometry, essentially there is enough weight on the arc $A = \{z : \theta' \leq \theta(z) \leq \theta''\}$ so that the expectation is non-zero. The conditional expectation $\mathbb{E}_\mu[z|A]$ is at least distance $\cos\Big(\frac{\theta''-\theta'}{2}\Big)$ away from 0 and the conditional expectation $\mathbb{E}_\mu[z|\mathbb{S}^1\backslash A]$ is at most distance 1 away from 0. □

The above proposition may be applied to obtaining concentration bounds by computing the expected angle $\mathbb{E}_\mu[\theta(z)]$ traversed. One may be able to obtain probability lower bounds such as 2.22 possibly by redefining the angle as traversed from an origin other than $(1, 0)$ (see also [14] for a similar problem).

The main idea here is to divide an absolutely convergent infinite sum $\sum_{n\geq 1} z_n$ into two parts $\sum_{n\in E} z_n$ and $\sum_{n\notin E} z_n$ and show that the two parts don't add up in some sense i.e. either the absolute values of the two parts are different; they are aligned at different angles; or one part bears higher weight on one half of a hyperplane than the other [15, 16, 17, 18, 19, 20]. We demonstrate this last idea as follows.

Suppose $\sum_{j\in J} z_j$ is a finite sum of vectors and suppose $q \neq 0$ is a hyperplane. Define $J^+ = \{j \in J : q.z_j > 0\}$ and $J^- = \{j \in J : q.z_j < 0\}$. Suppose that there exist sequences of pairwise disjoint sets $\{J_k^+\}_{k=1}^K \subseteq J^+$ and $\{J_k^-\}_{k=1}^K \subseteq J^-$ such that $\cup_{k=1}^K J_k^- = J^-$ and $q.(\sum_{j\in J_k^+} z_j + \sum_{j\in J_k^-} z_j) > 0$ for each $k \in \{1, ..., K\}$. Then, $\sum_{j\in J} z_j \neq 0$ since $q.(\sum_{j\in J} z_j) > 0$.

We prove the next proposition.

Proposition 2.6. *Let $\sigma' \in (0,1)$ be such that*

$$(1/2)^\sigma > (1/4)^\sigma + (1/7)^\sigma \text{ and } \frac{1 + \frac{\pi}{\sigma \ln(2)}}{8^\sigma} < 1. \tag{2.23}$$

for each $\sigma \in (\sigma', 1)$. Let $t = \frac{-\pi}{\ln(2)}$. Then, for each $\sigma \in (\sigma', 1)$, we have that $\zeta(\sigma, t) \neq 0$.

Proof. Let $s = (\sigma, t)$, where $\sigma \in (\sigma', 1)$. Define the following vectors in $\mathbb{R}^2$: $z_n := \frac{(-1)^{n+1}}{n^s}$ for $n \leq 7$ and define $z_8 := \sum_{n\geq 4} Z_n(s)$. Hence, $\zeta^*(s) = (\sum_{n=1}^7 z_n) + z_8$.

Now define the hyperplane $q = (1, 0)$. Then, we have as strictly positive the values $q.z_1, q.z_2, q.z_3, q.z_5, q.z_6 > 0$. We also have $q.z_4, q.z_7 < 0$. Suppose that $q.z_8 < 0$. Now define $J = \{1, 2, ..., 8\}$ and the collections $\{\{1, 2\}\}$ in J^+ and $\{\{4, 7, 8\}\}$ in J^-. Since $||z_8|| \leq 1$ (by applying the upper bound in proposition 2.3 and 2.23) we may then show that $q.(z_1 + z_2 + z_4 + z_7 + z_8) > 0$ by applying 2.23. If on the other hand we have $q.z_8 \geq 0$, then we may define the collections $\{\{2\}\}$ in J^+ and $\{\{4, 7\}\}$ in J^- and applying 2.23 show that $q.(z_2 + z_4 + z_7) > 0$. These facts may be proved geometrically by studying the defined points in $\mathbb{R}^2$ and the unit circle $\mathbb{S}^1$. ☐

3 Conclusion

In this paper, the methods presented establish non-trivial results, relative to the problem of resolving the Riemann Hypothesis. These correspond to the propositions 2.5 and the ideas presented prior to proposition 2.6. Such perspectives may allow us to prove further results pertaining to the Riemann Hypothesis.

Competing Interests

Author has declared that no competing interests exist.

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