



A Study of Generalized Fourth-Order Pell Sequences

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, we investigate the generalized fourth order Pell sequences and we deal with, in detail, three special cases which we call them as fourth order Pell, fourth order Pell-Lucas and modified fourth order Pell sequences.

Keywords: Pell numbers; fourth order Pell numbers; fourth order Pell-Lucas numbers.

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1 INTRODUCTION

In this paper, we introduce the generalized fourth order Pell sequences and we investigate, in detail, three special cases which we call them fourth order Pell, fourth order Pell-Lucas and modified fourth order Pell sequences.

It is well-known that the Pell sequence (sequence A000129 in [1]) $\{P_n\}$ is defined recursively by the equation, for $n \geq 0$

$$P_{n+2} = 2P_{n+1} + P_n$$

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in which $P_0 = 0$ and $P_1 = 1$. Next, we present the first few values of Pell numbers with positive and negative subscripts:

Table 1. The first few values of the Pell numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P_n	0	1	2	5	12	29	70	169	408	985	2378	5741	13860	33461	80782
P_{-n}	0	1	-2	5	-12	29	-70	169	-408	985	-2378	5741	-13860	33461	-80782

Pell sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [2,3,4,5,6,7,8,9,10,11]. For higher order Pell sequences, see [12,13,14,15].

A generalized Tetranacci sequence $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2, W_3; r_1, r_2, r_3, r_4)\}_{n \geq 0}$ is defined by the fourth-order recurrence relations

$$W_n = r_1 W_{n-1} + r_2 W_{n-2} + r_3 W_{n-3} + r_4 W_{n-4}, \quad W_0 = a, W_1 = b, W_2 = c, W_3 = d \quad (1.1)$$

where the initial values W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers and r_1, r_2, r_3, r_4 are real numbers.

Tetranacci sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [16,17,18,19,20,21,22].

The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r_3}{r_4} W_{-(n-1)} - \frac{r_2}{r_4} W_{-(n-2)} - \frac{r_1}{r_4} W_{-(n-3)} + \frac{1}{r_4} W_{-(n-4)}$$

for $n = 1, 2, 3, \dots$. Therefore, recurrence (1.1) holds for all integer n .

In this paper we consider the case $r_1 = 2, r_2 = r_3 = r_4 = 1$ and in this case we write $V_n = W_n$. A generalized fourth order Pell sequence $\{V_n\}_{n \geq 0} = \{V_n(V_0, V_1, V_2, V_3)\}_{n \geq 0}$ is defined by the fourth-order recurrence relations

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} \quad (1.2)$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3$ not all being zero.

The sequence $\{V_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} - 2V_{-(n-3)} + V_{-(n-4)}$$

for $n = 1, 2, 3, \dots$. Therefore, recurrence (1.2) holds for all integer n .

As $\{V_n\}$ is a fourth order recurrence sequence (difference equation), it's characteristic equation is

$$x^4 - 2x^3 - x^2 - x - 1 = 0. \quad (1.3)$$

The approximate value of the roots α, β, γ and δ of Equation (1.3) are given by

$$\begin{aligned} \alpha &= 2.592052792, \\ \beta &= -0.6631378984, \\ \gamma &= 0.03554255298 - 0.7619107877i, \\ \delta &= 0.03554255299 + 0.7619107877i. \end{aligned}$$

Note that we have the following identities:

$$\begin{aligned} \alpha + \beta + \gamma + \delta &= 2, \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -1, \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 1, \\ \alpha\beta\gamma\delta &= -1. \end{aligned}$$

The first few generalized fourth order Pell numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few generalized fourth order Pell numbers

n	V_n	V_{-n}
0	V_0	V_0
1	V_1	$-V_0 - V_1 - 2V_2 + V_3$
2	V_2	$-V_3 + 3V_2 - V_1$
3	V_3	$-2V_3 + 5V_2 + V_1 - V_0$
4	$2V_3 + V_2 + V_1 + V_0$	$V_3 - 4V_2 + 2V_1 + 4V_0$
5	$5V_3 + 3V_2 + 3V_1 + 2V_0$	$4V_3 - 9V_2 - 2V_1 - 4V_0$
6	$13V_3 + 8V_2 + 7V_1 + 5V_0$	$-2V_3 + 6V_2 - 3V_1 + 2V_0$
7	$34V_3 + 20V_2 + 18V_1 + 13V_0$	$-6V_3 + 16V_2 + 2V_1 - 7V_0$
8	$88V_3 + 52V_2 + 47V_1 + 34V_0$	$V_3 - 8V_2 + 7V_1 + 17V_0$
9	$262V_3 + 155V_2 + 140V_1 + 101V_0$	$13V_3 - 29V_2 - 5V_1 - 18V_0$
10	$693V_3 + 410V_2 + 370V_1 + 267V_0$	$-4V_3 + 11V_2 - 9V_1 + 17V_0$

Now we define three special case of the sequence $\{V_n\}$. Fourth-order Pell sequence $\{P_n^{(4)}\}_{n \geq 0}$, fourth-order Pell-Lucas sequence $\{Q_n^{(4)}\}_{n \geq 0}$ and modified fourth-order Pell sequence $\{E_n^{(4)}\}_{n \geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$P_{n+4}^{(4)} = 2P_{n+3}^{(4)} + P_{n+2}^{(4)} + P_{n+1}^{(4)} + P_n^{(4)}, \quad P_0^{(4)} = 0, P_1^{(4)} = 1, P_2^{(4)} = 2, P_3^{(4)} = 5, \quad (1.4)$$

and

$$Q_{n+4}^{(4)} = 2Q_{n+3}^{(4)} + Q_{n+2}^{(4)} + Q_{n+1}^{(4)} + Q_n^{(4)}, \quad Q_0^{(4)} = 4, Q_1^{(4)} = 2, Q_2^{(4)} = 6, Q_3^{(4)} = 17, \quad (1.5)$$

and

$$E_{n+4}^{(4)} = 2E_{n+3}^{(4)} + E_{n+2}^{(4)} + E_{n+1}^{(4)} + E_n^{(4)}, \quad E_0^{(4)} = 0, E_1^{(4)} = 1, E_2^{(4)} = 1, E_3^{(4)} = 3. \quad (1.6)$$

The sequences $\{P_n^{(4)}\}_{n \geq 0}$, $\{Q_n^{(4)}\}_{n \geq 0}$ and $\{E_n^{(4)}\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$P_{-n}^{(4)} = -P_{-(n-1)}^{(4)} - P_{-(n-2)}^{(4)} - 2P_{-(n-3)}^{(4)} + P_{-(n-4)}^{(4)} \quad (1.7)$$

and

$$Q_{-n}^{(4)} = -Q_{-(n-1)}^{(4)} - Q_{-(n-2)}^{(4)} - 2Q_{-(n-3)}^{(4)} + Q_{-(n-4)}^{(4)} \quad (1.8)$$

and

$$E_{-n}^{(4)} = -E_{-(n-1)}^{(4)} - E_{-(n-2)}^{(4)} - 2E_{-(n-3)}^{(4)} + E_{-(n-4)}^{(4)} \quad (1.9)$$

for $n = 1, 2, 3, \dots$ respectively. Therefore, recurrences (1.7), (1.8) and (1.9) hold for all integer n .

In the rest of the paper, for easy writing, we drop the superscripts and write P_n, Q_n and E_n for $P_n^{(4)}, Q_n^{(4)}$ and $E_n^{(4)}$, respectively.

Note that P_n is the sequence A103142 in [1], E_n is the sequence A190139 in [1] and Q_n sequence is't in the database of <http://oeis.org> [1], yet.

Next, we present the first few values of the fourth-order Pell, fourth-order Pell-Lucas and modified fourth-order Pell numbers with positive and negative subscripts:

Table 3. The first few values of the special fourth-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
P_n	0	1	2	5	13	34	88	228	591	1532	3971	10293	26680	69156
P_{-n}	0	0	0	1	-1	0	-1	4	-4	2	-7	17	-18	17
Q_n	4	2	6	17	46	117	303	786	2038	5282	13691	35488	91987	238435
Q_{-n}	4	-1	-1	-4	11	-6	2	-22	43	-31	34	-111	182	-170
E_n	0	1	1	3	8	21	54	140	363	941	2439	6322	16387	42476
E_{-n}	0	0	-1	2	-1	1	-5	8	-6	9	-24	35	-35	57

2 GENERATING FUNCTIONS

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence V_n .

Lemma 2.1. Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized fourth-order Pell sequence $\{V_n\}_{n \geq 0}$. Then, $\sum_{n=0}^{\infty} V_n x^n$ is given by

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2 + (V_3 - 2V_2 - V_1 - V_0)x^3}{1 - 2x - x^2 - x^3 - x^4}. \quad (2.1)$$

Proof. Using the definition of generalized fourth-order Pell numbers, and subtracting $2xf(x)$, $x^2f(x)$, $x^3f(x)$ and $x^4f(x)$ from $f(x)$ we obtain (note the shift in the index n in the third line)

$$\begin{aligned} & (1 - 2x - x^2 - x^3 - x^4)f_{V_n}(x) \\ &= \sum_{n=0}^{\infty} V_n x^n - 2x \sum_{n=0}^{\infty} V_n x^n - x^2 \sum_{n=0}^{\infty} V_n x^n - x^3 \sum_{n=0}^{\infty} V_n x^n - x^4 \sum_{n=0}^{\infty} V_n x^n \\ &= \sum_{n=0}^{\infty} V_n x^n - 2 \sum_{n=0}^{\infty} V_n x^{n+1} - \sum_{n=0}^{\infty} V_n x^{n+2} - \sum_{n=0}^{\infty} V_n x^{n+3} - \sum_{n=0}^{\infty} V_n x^{n+4} \\ &= \sum_{n=0}^{\infty} V_n x^n - 2 \sum_{n=1}^{\infty} V_{n-1} x^n - \sum_{n=2}^{\infty} V_{n-2} x^n - \sum_{n=3}^{\infty} V_{n-3} x^n - \sum_{n=4}^{\infty} V_{n-4} x^n \\ &= (V_0 + V_1 x + V_2 x^2 + V_3 x^3) - 2(V_0 x + V_1 x^2 + V_2 x^3) - (V_0 x^2 + V_1 x^3) - V_0 x^3 \\ &\quad + \sum_{n=4}^{\infty} (V_n - V_{n-1} - V_{n-2} - V_{n-3} - V_{n-4}) x^n \\ &= (V_0 + V_1 x + V_2 x^2 + V_3 x^3) - 2(V_0 x + V_1 x^2 + V_2 x^3) - (V_0 x^2 + V_1 x^3) - V_0 x^3 \\ &= V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2 + (V_3 - 2V_2 - V_1 - V_0)x^3 \end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2 + (V_3 - 2V_2 - V_1 - V_0)x^3}{1 - 2x - x^2 - x^3 - x^4}.$$

The previous Lemma gives the following results as particular examples.

Corollary 2.2. Generated functions of fourth-order Pell, Pell-Lucas and modified Pell numbers are

$$\sum_{n=0}^{\infty} P_n x^n = \frac{x}{1 - 2x - x^2 - x^3 - x^4},$$

and

$$\sum_{n=0}^{\infty} Q_n x^n = \frac{4 - 6x - 2x^2 - x^3}{1 - 2x - x^2 - x^3 - x^4},$$

and

$$\sum_{n=0}^{\infty} E_n x^n = \frac{x - x^2}{1 - 2x - x^2 - x^3 - x^4},$$

respectively.

3 OBTAINING BINET FORMULA FROM GENERATING FUNCTION

We next find Binet formula of generalized fourth order Pell numbers $\{V_n\}$ by the use of generating function for V_n .

Theorem 3.1. (Binet formula of generalized fourth order Pell numbers)

$$V_n = \frac{d_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{d_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{d_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{d_4 \delta^n}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)} \quad (3.1)$$

where

$$\begin{aligned} d_1 &= V_0 \alpha^3 + (V_1 - 2V_0) \alpha^2 + (V_2 - 2V_1 - V_0) \alpha + (V_3 - 2V_2 - V_1 - V_0), \\ d_2 &= V_0 \beta^3 + (V_1 - 2V_0) \beta^2 + (V_2 - 2V_1 - V_0) \beta + (V_3 - 2V_2 - V_1 - V_0), \\ d_3 &= V_0 \gamma^3 + (V_1 - 2V_0) \gamma^2 + (V_2 - 2V_1 - V_0) \gamma + (V_3 - 2V_2 - V_1 - V_0), \\ d_4 &= V_0 \delta^3 + (V_1 - 2V_0) \delta^2 + (V_2 - 2V_1 - V_0) \delta + (V_3 - 2V_2 - V_1 - V_0). \end{aligned}$$

Proof. Let

$$h(x) = 1 - 2x - x^2 - x^3 - x^4.$$

Then for some α, β, γ and δ we write

$$h(x) = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)$$

i.e.,

$$1 - 2x - x^2 - x^3 - x^4 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x). \quad (3.2)$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ ve $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α, β, γ and δ as the roots of

$$h\left(\frac{1}{x}\right) = 1 - \frac{2}{x} - \frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{x^4} = 0.$$

This implies $x^4 - x^3 - 2x^2 - x - 1 = 0$. Now, by (2.1) and (3.2), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2 + (V_3 - 2V_2 - V_1 - V_0)x^3}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)}.$$

Then we write

$$\frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)(1 - \delta x)} = \frac{A_1}{(1 - \alpha x)} + \frac{A_2}{(1 - \beta x)} + \frac{A_3}{(1 - \gamma x)} + \frac{A_4}{(1 - \delta x)}. \quad (3.3)$$

So

$$\begin{aligned} &V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - V_0)x^2 + (V_3 - 2V_2 - V_1 - V_0)x^3 \\ &= A_1(1 - \beta x)(1 - \gamma x)(1 - \delta x) + A_2(1 - \alpha x)(1 - \gamma x)(1 - \delta x) \\ &\quad + A_3(1 - \alpha x)(1 - \beta x)(1 - \delta x) + A_4(1 - \alpha x)(1 - \beta x)(1 - \gamma x). \end{aligned}$$

If we consider $x = \frac{1}{\alpha}$, we get $V_0 + (V_1 - 2V_0)\frac{1}{\alpha} + (V_2 - 2V_1 - V_0)\frac{1}{\alpha^2} + (V_3 - 2V_2 - V_1 - V_0)\frac{1}{\alpha^3} = A_1(1 - \frac{\beta}{\alpha})(1 - \frac{\gamma}{\alpha})(1 - \frac{\delta}{\alpha})$. This gives

$$\begin{aligned} A_1 &= \frac{\alpha^3(V_0 + (V_1 - 2V_0)\frac{1}{\alpha} + (V_2 - 2V_1 - V_0)\frac{1}{\alpha^2} + (V_3 - 2V_2 - V_1 - V_0)\frac{1}{\alpha^3})}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} \\ &= \frac{V_0\alpha^3 + (V_1 - 2V_0)\alpha^2 + (V_2 - 2V_1 - V_0)\alpha + (V_3 - 2V_2 - V_1 - V_0)}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} A_2 &= \frac{V_0\beta^3 + (V_1 - 2V_0)\beta^2 + (V_2 - 2V_1 - V_0)\beta + (V_3 - 2V_2 - V_1 - V_0)}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)}, \\ A_3 &= \frac{V_0\gamma^3 + (V_1 - 2V_0)\gamma^2 + (V_2 - 2V_1 - V_0)\gamma + (V_3 - 2V_2 - V_1 - V_0)}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)}, \\ A_4 &= \frac{V_0\delta^3 + (V_1 - 2V_0)\delta^2 + (V_2 - 2V_1 - V_0)\delta + (V_3 - 2V_2 - V_1 - V_0)}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}. \end{aligned}$$

Thus (3.3) can be written as

$$\sum_{n=0}^{\infty} V_n x^n = A_1(1 - \alpha x)^{-1} + A_2(1 - \beta x)^{-1} + A_3(1 - \gamma x)^{-1} + A_4(1 - \delta x)^{-1}.$$

This gives

$$\begin{aligned} \sum_{n=0}^{\infty} V_n x^n &= A_1 \sum_{n=0}^{\infty} \alpha^n x^n + A_2 \sum_{n=0}^{\infty} \beta^n x^n + A_3 \sum_{n=0}^{\infty} \gamma^n x^n + A_4 \sum_{n=0}^{\infty} \delta^n x^n \\ &= \sum_{n=0}^{\infty} (A_1 \alpha^n + A_2 \beta^n + A_3 \gamma^n + A_4 \delta^n) x^n. \end{aligned}$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n = A_1 \alpha^n + A_2 \beta^n + A_3 \gamma^n + A_4 \delta^n.$$

where

$$\begin{aligned} A_1 &= \frac{V_0\alpha^3 + (V_1 - 2V_0)\alpha^2 + (V_2 - 2V_1 - V_0)\alpha + (V_3 - 2V_2 - V_1 - V_0)}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)}, \\ A_2 &= \frac{V_0\beta^3 + (V_1 - 2V_0)\beta^2 + (V_2 - 2V_1 - V_0)\beta + (V_3 - 2V_2 - V_1 - V_0)}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)}, \\ A_3 &= \frac{V_0\gamma^3 + (V_1 - 2V_0)\gamma^2 + (V_2 - 2V_1 - V_0)\gamma + (V_3 - 2V_2 - V_1 - V_0)}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)}, \\ A_4 &= \frac{V_0\delta^3 + (V_1 - 2V_0)\delta^2 + (V_2 - 2V_1 - V_0)\delta + (V_3 - 2V_2 - V_1 - V_0)}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}, \end{aligned}$$

and then we get (3.1).

Next, using Theorem 3.1, we present the Binet formulas of fourth-order Pell, Pell-Lucas and modified Pell sequences.

Corollary 3.2. Binet formulas of fourth-order Pell, Pell-Lucas and modified Pell sequences are

$$P_n = \frac{\alpha^{n+2}}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{\beta^{n+2}}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{\gamma^{n+2}}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{\delta^{n+2}}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)},$$

and

$$Q_n = \alpha^n + \beta^n + \gamma^n + \delta^n,$$

and

$$E_n = \frac{(\alpha - 1)\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{(\beta - 1)\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{(\gamma - 1)\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{(\delta - 1)\delta^{n+1}}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)},$$

respectively.

Note that Binet formula of generalized fourth order Pell numbers can be represented as

$$V_n = \frac{\alpha d_1 \alpha^n}{2\alpha^3 + 2\alpha^2 + 3\alpha + 4} + \frac{\beta d_2 \beta^n}{2\beta^3 + 2\beta^2 + 3\beta + 4} + \frac{\gamma d_3 \gamma^n}{2\gamma^3 + 2\gamma^2 + 3\gamma + 4} + \frac{\delta d_4 \delta^n}{2\delta^3 + 2\delta^2 + 3\delta + 4} \quad (3.4)$$

which can be derived from a result ((4.20) in page 25) of Hanusa [23]. When we compare (3.1) and (3.4), we see the following identities:

$$\begin{aligned} \frac{1}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} &= \frac{\alpha}{2\alpha^3 + 2\alpha^2 + 3\alpha + 4}, \\ \frac{1}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} &= \frac{\beta}{2\beta^3 + 2\beta^2 + 3\beta + 4}, \\ \frac{1}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} &= \frac{\gamma}{2\gamma^3 + 2\gamma^2 + 3\gamma + 4}, \\ \frac{1}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)} &= \frac{\delta}{2\delta^3 + 2\delta^2 + 3\delta + 4}. \end{aligned}$$

Using the above identities, we can give the Binet formulas of fourth-order Pell, Pell-Lucas and modified Pell sequences in the following form: Binet formulas of fourth-order Pell, Pell-Lucas and modified Pell sequences are

$$P_n = \frac{\alpha^{n+3}}{2\alpha^3 + 2\alpha^2 + 3\alpha + 4} + \frac{\beta^{n+3}}{2\beta^3 + 2\beta^2 + 3\beta + 4} + \frac{\gamma^{n+3}}{2\gamma^3 + 2\gamma^2 + 3\gamma + 4} + \frac{\delta^{n+3}}{2\delta^3 + 2\delta^2 + 3\delta + 4},$$

and

$$Q_n = \alpha^n + \beta^n + \gamma^n + \delta^n,$$

and

$$E_n = \frac{(\alpha - 1)\alpha^{n+2}}{2\alpha^3 + 2\alpha^2 + 3\alpha + 4} + \frac{(\beta - 1)\beta^{n+2}}{2\beta^3 + 2\beta^2 + 3\beta + 4} + \frac{(\gamma - 1)\gamma^{n+2}}{2\gamma^3 + 2\gamma^2 + 3\gamma + 4} + \frac{(\delta - 1)\delta^{n+2}}{2\delta^3 + 2\delta^2 + 3\delta + 4}.$$

respectively.

We can also find Binet formulas by using matrix method which is given in [13]. Take $k = i = 4$ in Corollary 3.1 in [13]. Let

$$\begin{aligned} \Lambda &= \begin{pmatrix} \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \\ \delta^3 & \delta^2 & \delta & 1 \end{pmatrix}, \\ \Lambda_1 &= \begin{pmatrix} \alpha^{n-1} & \alpha^2 & \alpha & 1 \\ \beta^{n-1} & \beta^2 & \beta & 1 \\ \gamma^{n-1} & \gamma^2 & \gamma & 1 \\ \delta^{n-1} & \delta^2 & \delta & 1 \end{pmatrix}, \Lambda_2 = \begin{pmatrix} \alpha^3 & \alpha^{n-1} & \alpha & 1 \\ \beta^3 & \beta^{n-1} & \beta & 1 \\ \gamma^3 & \gamma^{n-1} & \gamma & 1 \\ \delta^3 & \delta^{n-1} & \delta & 1 \end{pmatrix}, \\ \Lambda_3 &= \begin{pmatrix} \alpha^3 & \alpha^2 & \alpha^{n-1} & 1 \\ \beta^3 & \beta^2 & \beta^{n-1} & 1 \\ \gamma^3 & \gamma^2 & \gamma^{n-1} & 1 \\ \delta^3 & \delta^2 & \delta^{n-1} & 1 \end{pmatrix}, \Lambda_4 = \begin{pmatrix} \alpha^3 & \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^3 & \beta^2 & \beta & \beta^{n-1} \\ \gamma^3 & \gamma^2 & \gamma & \gamma^{n-1} \\ \delta^3 & \delta^2 & \delta & \delta^{n-1} \end{pmatrix}. \end{aligned}$$

Then the Binet formula for fourth-order Pell numbers is

$$\begin{aligned}
 P_n &= \frac{1}{\det(\Lambda)} \sum_{j=1}^4 P_{5-j} \det(\Lambda_j) = \frac{1}{\Lambda} (P_4 \det(\Lambda_1) + P_3 \det(\Lambda_2) + P_2 \det(\Lambda_3) + P_1 \det(\Lambda_4)) \\
 &= \frac{1}{\det(\Lambda)} (13 \det(\Lambda_1) + 5 \det(\Lambda_2) + 2 \det(\Lambda_3) + \det(\Lambda_4)) \\
 &= \frac{\alpha^{n+2}}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{\beta^{n+2}}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{\gamma^{n+2}}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{\delta^{n+2}}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}.
 \end{aligned}$$

Similarly, we obtain the Binet formula for fourth-order Pell-Lucas and modified fourth-order Pell numbers as

$$\begin{aligned}
 Q_n &= \frac{1}{\Lambda} (Q_4 \det(\Lambda_1) + Q_3 \det(\Lambda_2) + Q_2 \det(\Lambda_3) + Q_1 \det(\Lambda_4)) \\
 &= \frac{1}{\Lambda} (46 \det(\Lambda_1) + 17 \det(\Lambda_2) + 6 \det(\Lambda_3) + 2 \det(\Lambda_4)) \\
 &= \alpha^n + \beta^n + \gamma^n + \delta^n
 \end{aligned}$$

and

$$\begin{aligned}
 E_n &= \frac{1}{\Lambda} (E_4 \det(\Lambda_1) + E_3 \det(\Lambda_2) + E_2 \det(\Lambda_3) + E_1 \det(\Lambda_4)) \\
 &= \frac{1}{\Lambda} (8 \det(\Lambda_1) + 3 \det(\Lambda_2) + \det(\Lambda_3) + \det(\Lambda_4))
 \end{aligned}$$

respectively.

4 SIMSON FORMULAS

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following Theorem gives generalization of this result to the generalized Tetranacci sequence $\{W_n\}$.

Theorem 4.1 (Simson Formula of Generalized Tetranacci Numbers). *For all integers n we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n r_4^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \quad (4.1)$$

Proof. (4.1) is given in Soykan [24].

A special case of the above theorem is the following Theorem which gives Simson formula of the generalized fourth-order Pell sequence $\{V_n\}$.

Theorem 4.2 (Simson Formula of Generalized Fourth-Order Pell Numbers). *For all integers n we have*

$$\begin{vmatrix} V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} \end{vmatrix} = (-1)^n \begin{vmatrix} V_3 & V_2 & V_1 & V_0 \\ V_2 & V_1 & V_0 & V_{-1} \\ V_1 & V_0 & V_{-1} & V_{-2} \\ V_0 & V_{-1} & V_{-2} & V_{-3} \end{vmatrix}.$$

The previous Theorem gives the following results as particular examples.

Corollary 4.3. *Simson formula of fourth-order Pell, Pell-Lucas and modified Pell numbers are given as*

$$\begin{vmatrix} P_{n+3} & P_{n+2} & P_{n+1} & P_n \\ P_{n+2} & P_{n+1} & P_n & P_{n-1} \\ P_{n+1} & P_n & P_{n-1} & P_{n-2} \\ P_n & P_{n-1} & P_{n-2} & P_{n-3} \end{vmatrix} = -(-1)^n = (-1)^{n+1},$$

and

$$\begin{vmatrix} Q_{n+3} & Q_{n+2} & Q_{n+1} & Q_n \\ Q_{n+2} & Q_{n+1} & Q_n & Q_{n-1} \\ Q_{n+1} & Q_n & Q_{n-1} & Q_{n-2} \\ Q_n & Q_{n-1} & Q_{n-2} & Q_{n-3} \end{vmatrix} = 1423(-1)^n,$$

and

$$\begin{vmatrix} E_{n+3} & E_{n+2} & E_{n+1} & E_n \\ E_{n+2} & E_{n+1} & E_n & E_{n-1} \\ E_{n+1} & E_n & E_{n-1} & E_{n-2} \\ E_n & E_{n-1} & E_{n-2} & E_{n-3} \end{vmatrix} = -4(-1)^n = 4(-1)^{n+1},$$

respectively.

5 SOME IDENTITIES

In this section, we obtain some identities of fourth order Pell, fourth order Pell-Lucas and modified fourth order Pell numbers. First, we can give a few basic relations between $\{P_n\}$ and $\{Q_n\}$.

Lemma 5.1. *The following equalities are true:*

$$Q_n = 11P_{n+5} - 26P_{n+4} - 4P_{n+3} - 6P_{n+2}, \tag{5.1}$$

$$Q_n = -4P_{n+4} + 7P_{n+3} + 5P_{n+2} + 11P_{n+1}, \tag{5.2}$$

$$Q_n = -P_{n+3} + P_{n+2} + 7P_{n+1} - 4P_n, \tag{5.3}$$

$$Q_n = -P_{n+2} + 6P_{n+1} - 5P_n - P_{n-1}, \tag{5.4}$$

$$Q_n = 4P_{n+1} - 6P_n - 2P_{n-1} - P_{n-2}, \tag{5.5}$$

and

$$1423P_n = 44Q_{n+5} - 270Q_{n+4} + 426Q_{n+3} + 5Q_{n+2}, \tag{5.6}$$

$$1423P_n = -182Q_{n+4} + 470Q_{n+3} + 49Q_{n+2} + 44Q_{n+1}, \tag{5.7}$$

$$1423P_n = 106Q_{n+3} - 133Q_{n+2} - 138Q_{n+1} - 182Q_n, \tag{5.8}$$

$$1423P_n = 79Q_{n+2} - 32Q_{n+1} - 76Q_n + 106Q_{n-1}, \tag{5.9}$$

$$1423P_n = 126Q_{n+1} + 3Q_n + 185Q_{n-1} + 79Q_{n-2} \tag{5.10}$$

Proof. Note that all the identities hold for all integers n . We prove (5.1). To show (5.1), writing

$$Q_n = a \times P_{n+5} + b \times P_{n+4} + c \times P_{n+3} + d \times P_{n+2}$$

and solving the system of equations

$$\begin{aligned} Q_0 &= a \times P_5 + b \times P_4 + c \times P_3 + d \times P_2 \\ Q_1 &= a \times P_6 + b \times P_5 + c \times P_4 + d \times P_3 \\ Q_2 &= a \times P_7 + b \times P_6 + c \times P_5 + d \times P_4 \\ Q_3 &= a \times P_8 + b \times P_7 + c \times P_6 + d \times P_5 \end{aligned}$$

we find that $a = 11, b = -26, c = -4, d = -6$. The other equalities can be proved similarly.

Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we present a few basic relations between $\{P_n\}$ and $\{E_n\}$.

Lemma 5.2. *The following equalities are true:*

$$\begin{aligned} E_n &= -P_{n+5} + 4P_{n+4} - 4P_{n+3} + P_{n+2}, \\ E_n &= 2P_{n+4} - 5P_{n+3} - P_{n+1}, \\ E_n &= -P_{n+3} + 2P_{n+2} + P_{n+1} + 2P_n, \\ E_n &= P_n - P_{n-1} \end{aligned}$$

and

$$\begin{aligned} 4P_n &= -3E_{n+5} + 7E_{n+4} + 2E_{n+3} + E_{n+2}, \\ 4P_n &= E_{n+4} - E_{n+3} - 2E_{n+2} - 3E_{n+1}, \\ 4P_n &= E_{n+3} - E_{n+2} - 2E_{n+1} + E_n, \\ 4P_n &= E_{n+2} - E_{n+1} + 2E_n + 1E_{n-1}. \end{aligned}$$

Thirdly, we give a few basic relations between $\{Q_n\}$ and $\{E_n\}$.

Lemma 5.3. *The following equalities are true:*

$$\begin{aligned} 4Q_n &= 19E_{n+5} - 35E_{n+4} - 26E_{n+3} - 25E_{n+2}, \\ 4Q_n &= 3E_{n+4} - 7E_{n+3} - 6E_{n+2} + 19E_{n+1}, \\ 4Q_n &= -E_{n+3} - 3E_{n+2} + 22E_{n+1} + 3E_n, \\ 4Q_n &= -5E_{n+2} + 21E_{n+1} + 2E_n - E_{n-1}, \\ 4Q_n &= 11E_{n+1} - 3E_n - 6E_{n-1} - 5E_{n-2}, \end{aligned}$$

and

$$\begin{aligned} 1423E_n &= 39Q_{n+5} - 304Q_{n+4} + 701Q_{n+3} - 416Q_{n+2}, \\ 1423E_n &= -226Q_{n+4} + 740Q_{n+3} - 377Q_{n+2} + 39Q_{n+1}, \\ 1423E_n &= 288Q_{n+3} - 603Q_{n+2} - 187Q_{n+1} - 226Q_n, \\ 1423E_n &= -27Q_{n+2} + 101Q_{n+1} + 62Q_n + 288Q_{n-1}, \\ 1423E_n &= 47Q_{n+1} + 35Q_n + 261Q_{n-1} - 27Q_{n-2}. \end{aligned}$$

We now present a few special identities for the modified fourth order Pell sequence $\{E_n\}$.

Theorem 5.4. (Catalan's identity) For all natural numbers n and m , the following identity holds

$$E_{n+m}E_{n-m} - E_n^2 = (P_{n+m} - P_{n+m-1})(P_{n-m} - P_{n-m-1}) - (P_n - P_{n-1})^2.$$

Proof. We use the identity

$$E_n = P_n - P_{n-1}.$$

Note that for $m = 1$ in Catalan's identity, we get the Cassini identity for the modified fourth order Pell sequence.

Corollary 5.5. (Cassini's identity) For all natural numbers n and m , the following identity holds

$$E_{n+1}E_{n-1} - E_n^2 = (P_{n+1} - P_n)(P_{n-1} - P_{n-2}) - (P_n - P_{n-1})^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using $E_n = P_n - P_{n-1}$. The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of modified fourth order Pell sequence $\{E_n\}$.

Theorem 5.6. Let n and m be any integers. Then the following identities are true:

(a) (d'Ocagne's identity)

$$E_{m+1}E_n - E_mE_{n+1} = (P_{m+1} - P_m)(P_n - P_{n-1}) - (P_m - P_{m-1})(P_{n+1} - P_n).$$

(b) (Gelin-Cesàro's identity)

$$E_{n+2}E_{n+1}E_{n-1}E_{n-2} - E_n^4 = (P_{n+2} - P_{n+1})(P_{n+1} - P_n)(P_{n-1} - P_{n-2})(P_{n-2} - P_{n-3}) - (P_n - P_{n-1})^4.$$

(c) (Melham's identity)

$$E_{n+1}E_{n+2}E_{n+6} - E_{n+3}^3 = (P_{n+1} - P_n)(P_{n+2} - P_{n+1})(P_{n+6} - P_{n+5}) - (P_{n+3} - P_{n+2})^3.$$

Proof. Use the identity $E_n = P_n - P_{n-1}$.

6 LINEAR SUMS

The following Theorem presents summing formulas of generalized fourth order Pell numbers.

Theorem 6.1. For $n \geq 0$ we have the following formulas:

(a) (Sum of the generalized fourth order Pell numbers)

$$\sum_{k=0}^n V_k = \frac{1}{4}(V_{n+4} - V_{n+3} - 2V_{n+2} - 3V_{n+1} - V_3 + V_2 + 2V_1 + 3V_0).$$

(b)

$$\sum_{k=0}^n V_{2k} = \frac{1}{8}(-V_{2n+2} + 5V_{2n+1} + 2V_{2n} + 3V_{2n-1} - 3V_3 + 7V_2 - 2V_1 + 9V_0).$$

(c)

$$\sum_{k=0}^n V_{2k+1} = \frac{1}{8}(3V_{2n+2} + V_{2n+1} + 2V_{2n} - V_{2n-1} + V_3 - 5V_2 + 6V_1 - 3V_0).$$

Proof.

(a) Using the recurrence relation

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4}$$

i.e.

$$V_{n-3} = V_n - 2V_{n-1} - V_{n-2}$$

we obtain

$$\begin{aligned} V_0 &= V_4 - 2V_3 - V_2 - V_1 \\ V_1 &= V_5 - 2V_4 - V_3 - V_2 \\ V_2 &= V_6 - 2V_5 - V_4 - V_3 \\ V_3 &= V_7 - 2V_6 - V_5 - V_4 \\ V_4 &= V_8 - 2V_7 - V_6 - V_5 \\ &\vdots \\ V_{n-4} &= V_n - 2V_{n-1} - V_{n-2} - V_{n-3} \\ V_{n-3} &= V_{n+1} - 2V_n - V_{n-1} - V_{n-2} \\ V_{n-2} &= V_{n+2} - 2V_{n+1} - V_n - V_{n-1} \\ V_{n-1} &= V_{n+3} - 2V_{n+2} - V_{n+1} - V_n \\ V_n &= V_{n+4} - 2V_{n+3} - V_{n+2} - V_{n+1}. \end{aligned}$$

If we add the equations by side by, we get

$$\begin{aligned} \sum_{k=0}^n V_k &= (V_{n+4} + V_{n+3} + V_{n+2} + V_{n+1} - V_3 - V_2 - V_1 - V_0 + \sum_{k=0}^n V_k) \\ &\quad - 2(V_{n+3} + V_{n+2} + V_{n+1} - V_2 - V_1 - V_0 + \sum_{k=0}^n V_k) - (V_{n+2} + V_{n+1} - V_1 - V_0 + \sum_{k=0}^n V_k) \\ &\quad - (V_{n+1} - V_0 + \sum_{k=0}^n V_k). \end{aligned}$$

Then, solving the above equality we obtain

$$\sum_{k=0}^n V_k = \frac{1}{4}(V_{n+4} - V_{n+3} - 2V_{n+2} - 3V_{n+1} - V_3 + V_2 + 2V_1 + 3V_0).$$

(b) and (c) Using the recurrence relation

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4}$$

i.e.

$$2V_{n-1} = V_n - V_{n-2} - V_{n-3} - V_{n-4}$$

we obtain

$$\begin{aligned} 2V_3 &= V_4 - V_2 - V_1 - V_0 \\ 2V_5 &= V_6 - V_4 - V_3 - V_2 \\ 2V_7 &= V_8 - V_6 - V_5 - V_4 \\ &\vdots \\ 2V_{2n-1} &= V_{2n} - V_{2n-2} - V_{2n-3} - V_{2n-4} \\ 2V_{2n+1} &= V_{2n+2} - V_{2n} - V_{2n-1} - V_{2n-2} \\ 2V_{2n+3} &= V_{2n+4} - V_{2n+2} - V_{2n+1} - V_{2n}. \end{aligned}$$

Now, if we add the above equations by side by, we get

$$2(-V_1 + \sum_{k=0}^n V_{2k+1}) = (V_{2n+2} - V_2 - V_0 + \sum_{k=0}^n V_{2k}) - (-V_0 + \sum_{k=0}^n V_{2k}) - (-V_{2n+1} + \sum_{k=0}^n V_{2k+1}) - (-V_{2n} + \sum_{k=0}^n V_{2k}).$$

Similarly, using the recurrence relation

$$V_n = 2V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4}$$

i.e.

$$2V_{n-1} = V_n - V_{n-2} - V_{n-3} - V_{n-4}$$

we write the following obvious equations;

$$\begin{aligned} 2V_2 &= V_3 - V_1 - V_0 - V_{-1} \\ 2V_4 &= V_5 - V_3 - V_2 - V_1 \\ 2V_6 &= V_7 - V_5 - V_4 - V_3 \\ 2V_8 &= V_9 - V_7 - V_6 - V_5 \\ &\vdots \\ 2V_{2n-2} &= V_{2n-1} - V_{2n-3} - V_{2n-4} - V_{2n-5} \\ 2V_{2n} &= V_{2n+1} - V_{2n-1} - V_{2n-2} - V_{2n-3} \\ 2V_{2n+2} &= V_{2n+3} - V_{2n+1} - V_{2n} - V_{2n-1} \\ 2V_{2n+4} &= V_{2n+5} - V_{2n+3} - V_{2n+2} - V_{2n+1} \end{aligned}$$

Now, if we add the above equations by side by, we obtain

$$\begin{aligned} 2(-V_0 + \sum_{k=0}^n V_{2k}) &= (-V_1 + \sum_{k=0}^n V_{2k+1}) - (-V_{2n+1} + \sum_{k=0}^n V_{2k+1}) - (-V_{2n} + \sum_{k=0}^n V_{2k}) \\ &\quad - (-V_{2n+1} - V_{2n-1} + V_{-1} + \sum_{k=0}^n V_{2k+1}). \end{aligned}$$

Then, solving the following system

$$\begin{aligned} 2(-V_1 + \sum_{k=0}^n V_{2k+1}) &= (V_{2n+2} - V_2 - V_0 + \sum_{k=0}^n V_{2k}) - (-V_0 + \sum_{k=0}^n V_{2k}) - (-V_{2n+1} + \sum_{k=0}^n V_{2k+1}) \\ &\quad - (-V_{2n} + \sum_{k=0}^n V_{2k}), \\ 2(-V_0 + \sum_{k=0}^n V_{2k}) &= (-V_1 + \sum_{k=0}^n V_{2k+1}) - (-V_{2n+1} + \sum_{k=0}^n V_{2k+1}) - (-V_{2n} + \sum_{k=0}^n V_{2k}) \\ &\quad - (-V_{2n+1} - V_{2n-1} + (-V_0 - V_1 - 2V_2 + V_3) + \sum_{k=0}^n V_{2k+1}), \end{aligned}$$

where

$$V_{-1} = -V_0 - V_1 - 2V_2 + V_3$$

the required result of (b) and (c) follow.

As special cases of above Theorem, we have the following three Corollaries. First one presents some summing formulas of fourth order Pell numbers.

Corollary 6.2. For $n \geq 0$ we have the following formulas:

(a) (Sum of the fourth order Pell numbers)

$$\sum_{k=0}^n P_k = \frac{1}{4}(P_{n+4} - P_{n+3} - 2P_{n+2} - 3P_{n+1} - 1).$$

(b) $\sum_{k=0}^n P_{2k} = \frac{1}{8}(-P_{2n+2} + 5P_{2n+1} + 2P_{2n} + 3P_{2n-1} - 3).$

(c) $\sum_{k=0}^n P_{2k+1} = \frac{1}{8}(3P_{2n+2} + P_{2n+1} + 2P_{2n} - P_{2n-1} + 1).$

Secon one presents some summing formulas of fourth order Pell-Lucas numbers.

Corollary 6.3. For $n \geq 0$ we have the following formulas:

(a) (Sum of the fourth order Pell-Lucas numbers)

$$\sum_{k=0}^n Q_k = \frac{1}{4}(Q_{n+4} - Q_{n+3} - 2Q_{n+2} - 3Q_{n+1} + 5).$$

(b) $\sum_{k=0}^n Q_{2k} = \frac{1}{8}(-Q_{2n+2} + 5Q_{2n+1} + 2Q_{2n} + 3Q_{2n-1} + 23).$

(c) $\sum_{k=0}^n Q_{2k+1} = \frac{1}{8}(3Q_{2n+2} + Q_{2n+1} + 2Q_{2n} - Q_{2n-1} - 13).$

Last one presents some summing formulas of modified fourth order Pell numbers.

Corollary 6.4. For $n \geq 0$ we have the following formulas:

(a) (Sum of the modified fourth order Pell numbers)

$$\sum_{k=0}^n E_k = \frac{1}{4}(E_{n+4} - E_{n+3} - 2E_{n+2} - 3E_{n+1}).$$

(b) $\sum_{k=0}^n E_{2k} = \frac{1}{8}(-E_{2n+2} + 5E_{2n+1} + 2E_{2n} + 3E_{2n-1} - 4).$

(c) $\sum_{k=0}^n E_{2k+1} = \frac{1}{8}(3E_{2n+2} + E_{2n+1} + 2E_{2n} - E_{2n-1} + 4).$

7 MATRICES RELATED WITH GENERALIZED FOURTH-ORDER PELL NUMBERS

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r_1 & r_2 & r_3 & r_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{7.1}$$

For matrix formulation (7.1), see [25]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \\ W_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ r_1 & r_2 & r_3 & r_4 \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{pmatrix}.$$

We define the square matrix A of order 4 as:

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det M = -1$. From (1.2) we have

$$\begin{pmatrix} V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix}. \quad (7.2)$$

and from (7.1) (or using (7.2) and induction) we have

$$\begin{pmatrix} V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V = P$ in (7.2) we have

$$\begin{pmatrix} P_{n+3} \\ P_{n+2} \\ P_{n+1} \\ P_n \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} P_{n+2} \\ P_{n+1} \\ P_n \\ P_{n-1} \end{pmatrix}. \quad (7.3)$$

We also define

$$B_n = \begin{pmatrix} P_{n+1} & P_n + P_{n-1} + P_{n-2} & P_n + P_{n-1} & P_n \\ P_n & P_{n-1} + P_{n-2} + P_{n-3} & P_{n-1} + P_{n-2} & P_{n-1} \\ P_{n-1} & P_{n-2} + P_{n-3} + P_{n-4} & P_{n-2} + P_{n-3} & P_{n-2} \\ P_{n-2} & P_{n-3} + P_{n-4} + P_{n-5} & P_{n-3} + P_{n-4} & P_{n-3} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & V_n + V_{n-1} + V_{n-2} & V_n + V_{n-1} & V_n \\ V_n & V_{n-1} + V_{n-2} + V_{n-3} & V_{n-1} + V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-2} + V_{n-3} + V_{n-4} & V_{n-2} + V_{n-3} & V_{n-2} \\ V_{n-2} & V_{n-3} + V_{n-4} + V_{n-5} & V_{n-3} + V_{n-4} & V_{n-3} \end{pmatrix}.$$

Theorem 7.1. For all integer $m, n \geq 0$, we have

- (a) $B_n = A^n$
- (b) $C_1 A^n = A^n C_1$
- (c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

- (a) By expanding the vectors on the both sides of (7.3) to 4-columns and multiplying the obtained on the right-hand side by A , we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1} B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

- (b) Using (a) and definition of C_1 , (b) follows.

(c) We have

$$\begin{aligned}
 AC_{n-1} &= \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_n & V_{n-1} + V_{n-2} + V_{n-3} & V_{n-1} + V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-2} + V_{n-3} + V_{n-4} & V_{n-2} + V_{n-3} & V_{n-2} \\ V_{n-2} & V_{n-3} + V_{n-4} + V_{n-5} & V_{n-3} + V_{n-4} & V_{n-3} \\ V_{n-3} & V_{n-4} + V_{n-5} + V_{n-6} & V_{n-4} + V_{n-5} & V_{n-4} \end{pmatrix} \\
 &= \begin{pmatrix} V_{n+1} & V_n + V_{n-1} + V_{n-2} & V_n + V_{n-1} & V_n \\ V_n & V_{n-1} + V_{n-2} + V_{n-3} & V_{n-1} + V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-2} + V_{n-3} + V_{n-4} & V_{n-2} + V_{n-3} & V_{n-2} \\ V_{n-2} & V_{n-3} + V_{n-4} + V_{n-5} & V_{n-3} + V_{n-4} & V_{n-3} \end{pmatrix} = C_n.
 \end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now we obtain

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and similarly

$$C_{n+m} = B_mC_n.$$

Some properties of A^n matrix can be given as

$$A^n = 2A^{n-1} + A^{n-2} + A^{n-3} + A^{n-4}$$

and

$$A^{n+m} = A^nA^m = A^mA^n$$

for all integer m and n .

Theorem 7.2. For $m, n \geq 0$ we have

$$\begin{aligned}
 V_{n+m} &= V_nP_{m+1} + V_{n-1}(P_m + P_{m-1} + P_{m-2}) + V_{n-2}(P_m + P_{m-1}) + V_{n-3}P_m \quad (7.4) \\
 &= V_nP_{m+1} + (V_{n-1} + V_{n-2} + V_{n-3})P_m + (V_{n-1} + V_{n-2})P_{m-1} + V_{n-1}P_{m-2}.
 \end{aligned}$$

Proof. From the equation $C_{n+m} = C_nB_m = B_mC_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m . From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m . We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and C_nB_m . This completes the proof.

Remark 7.1. By induction, it can be proved that for all integers $m, n \leq 0$, (7.4) holds. So for all integers m, n (7.4) is true.

Corollary 7.3. For all integers m, n , we have

$$\begin{aligned}
 P_{n+m} &= P_nP_{m+1} + P_{n-1}(P_m + P_{m-1} + P_{m-2}) + P_{n-2}(P_m + P_{m-1}) + P_{n-3}P_m, \\
 Q_{n+m} &= Q_nP_{m+1} + Q_{n-1}(P_m + P_{m-1} + P_{m-2}) + Q_{n-2}(P_m + P_{m-1}) + Q_{n-3}P_m, \\
 E_{n+m} &= E_nP_{m+1} + E_{n-1}(P_m + P_{m-1} + P_{m-2}) + E_{n-2}(P_m + P_{m-1}) + E_{n-3}P_m.
 \end{aligned}$$

8 CONCLUSION

In the literature, there have been so many studies of the sequences of numbers and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. We introduce the generalized fourth order Pell sequences and we present Binet's formulas, generating functions, Simson formulas, the summation formulas, some identities and matrices for these sequences.

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COMPETING INTERESTS

Authors has declared that no competing interests exist.

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